

微分几何习题解答

习题一

1. 设 $\mathcal{T}$ 是 E^3 的一个(双射)变换. 假设对于 E^3 的某个开子集 U , $\mathcal{T}(U) = U$ 并且 $\mathcal{T}$ 保持 U 中任意两点的距离. 证明 $\mathcal{T}$ 是一个合同变换.

证明: 经适当修改, 与定理 2.1 的第二个证明基本相同. (须假设 $\mathcal{T}$ 是 C^2 的.)

2. 设 $\mathbf{a}(t)$ 是连续可微的向量值函数, 证明:

(1) $|\mathbf{a}(t)|$ 是常数当且仅当 $\langle \mathbf{a}(t), \mathbf{a}'(t) \rangle = 0$;

(2) 假设 $|\mathbf{a}(t)| \neq 0$. 则 $\mathbf{a}(t)$ 的方向不变当且仅当 $\mathbf{a}(t) \wedge \mathbf{a}'(t) = \mathbf{0}$.

证明:

(1) $|\mathbf{a}(t)| = \text{常数} \Leftrightarrow |\mathbf{a}(t)|^2 = \langle \mathbf{a}(t), \mathbf{a}(t) \rangle = \text{常数} \Leftrightarrow$

$\langle \mathbf{a}(t), \mathbf{a}(t) \rangle' = 2\langle \mathbf{a}(t), \mathbf{a}'(t) \rangle = 0 \Leftrightarrow \langle \mathbf{a}(t), \mathbf{a}'(t) \rangle = 0$.

(2) 由 $|\mathbf{a}(t)| \neq 0$, $\mathbf{a}(t)$ 的方向不变 $\Leftrightarrow \mathbf{e}(t) := \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|}$ 是单位常向量 $\Leftrightarrow \mathbf{e}'(t) = \mathbf{0}$. 而由 $\langle \mathbf{e}(t), \mathbf{e}(t) \rangle = 1$, 有 $\langle \mathbf{e}(t), \mathbf{e}'(t) \rangle = 0$, 即: $\mathbf{e}(t) \perp \mathbf{e}'(t)$. 因此, $\mathbf{e}'(t) = \mathbf{0} \Leftrightarrow \mathbf{e}(t) // \mathbf{e}'(t) \Leftrightarrow \mathbf{0} = \mathbf{e}(t) \wedge \mathbf{e}'(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|} \wedge (\frac{\mathbf{a}'(t)}{|\mathbf{a}(t)|} + (\frac{1}{|\mathbf{a}(t)|})' \mathbf{a}(t)) = \mathbf{a}(t) \wedge \mathbf{a}'(t)$.

3. 验证性质 1.1 和性质 1.2.

性质 1.1 设 $\mathbf{v}_i$ ($1 \leq i \leq 4$) 是 $\mathbb{R}^3$ 的四个向量.

(1) $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \mathbf{v}_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \mathbf{v}_3$;

(2) Lagrange 恒等式:

$$\langle \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v}_3 \wedge \mathbf{v}_4 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \langle \mathbf{v}_2, \mathbf{v}_4 \rangle - \langle \mathbf{v}_1, \mathbf{v}_4 \rangle \langle \mathbf{v}_2, \mathbf{v}_3 \rangle;$$

(3) $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) = (\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2)$.

证明: 设 $\mathbf{v}_i = (x^i, y^i, z^i)$ ($1 \leq i \leq 4$).

(1) 由定义, 计算有

$$\begin{aligned} \mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) &= (x^1, y^1, z^1) \wedge \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^1 & y^1 & z^1 \\ y^2 z^3 - y^3 z^2 & -x^2 z^3 + x^3 z^2 & x^2 y^3 - x^3 y^2 \end{vmatrix} \\ &= (x^2 y^1 y^3 - x^3 y^1 y^2 + x^2 z^1 z^3 - x^3 z^1 z^2, -x^1 x^2 y^3 + x^1 x^3 y^2 + y^2 z^1 z^3 - y^3 z^1 z^2, \\ &\quad -x^1 x^2 z^3 + x^1 x^3 z^2 - y^1 y^2 z^3 + y^1 y^3 z^2) \\ &= ((x^1 x^3 + y^1 y^3 + z^1 z^3)x^2 - (x^1 x^2 + y^1 y^2 + z^1 z^2)x^3, \\ &\quad (x^1 x^3 + y^1 y^3 + z^1 z^3)y^2 - (x^1 x^2 + y^1 y^2 + z^1 z^2)y^3, \\ &\quad (x^1 x^3 + y^1 y^3 + z^1 z^3)z^2 - (x^1 x^2 + y^1 y^2 + z^1 z^2)z^3) \\ &= \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \mathbf{v}_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \mathbf{v}_3. \end{aligned}$$

(2) 由定义, 计算有

$$\begin{aligned}
 \langle \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v}_3 \wedge \mathbf{v}_4 \rangle &= \left\langle \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^1 & y^1 & z^1 \\ x^2 & y^2 & z^2 \end{vmatrix}, \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^3 & y^3 & z^3 \\ x^4 & y^4 & z^4 \end{vmatrix} \right\rangle \\
 &= \langle (y^1 z^2 - y^2 z^1, -x^1 z^2 + x^2 z^1, x^1 y^2 - x^2 y^1), (y^3 z^4 - y^4 z^3, -x^3 z^4 + x^4 z^3, x^3 y^4 - x^4 y^3) \rangle \\
 &= y^1 y^3 z^2 z^4 - y^1 y^4 z^2 z^3 - y^2 y^3 z^1 z^4 + y^2 y^4 z^1 z^3 + x^1 x^3 z^2 z^4 - x^1 x^4 z^2 z^3 \\
 &\quad - x^2 x^3 z^1 z^4 + x^2 x^4 z^1 z^3 + x^1 x^3 y^2 y^4 - x^1 x^4 y^2 y^3 - x^2 x^3 y^1 y^4 + x^2 x^4 y^1 y^3 \\
 &= (x^1 x^3 + y^1 y^3 + z^1 z^3)(x^2 x^4 + y^2 y^4 + z^2 z^4) - (x^1 x^4 + y^1 y^4 + z^1 z^4)(x^2 x^3 + y^2 y^3 + z^2 z^4) \\
 &= \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \langle \mathbf{v}_2, \mathbf{v}_4 \rangle - \langle \mathbf{v}_1, \mathbf{v}_4 \rangle \langle \mathbf{v}_2, \mathbf{v}_3 \rangle.
 \end{aligned}$$

(3) 由定义, 计算有

$$\begin{aligned}
 (\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) &= \langle \mathbf{v}_2, \mathbf{v}_3 \wedge \mathbf{v}_1 \rangle = \langle (x^2, y^2, z^2), \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^3 & y^3 & z^3 \\ x^1 & y^1 & z^1 \end{vmatrix} \rangle \\
 &= \langle (x^2, y^2, z^2), (y^3 z^1 - y^1 z^3, -x^3 z^1 + x^1 z^3, x^3 y^1 - x^1 y^3) \rangle \\
 &= x^2 y^3 z^1 - x^2 y^1 z^3 - x^3 y^2 z^1 + x^1 y^2 z^3 + x^3 y^1 z^2 - x^1 y^3 z^2 \\
 &= x^1(y^2 z^3 - y^3 z^2) + y^1(-x^2 z^3 + x^3 z^2) + z^1(x^2 y^3 - x^3 y^2) \\
 &= \langle \mathbf{v}_1, \mathbf{v}_2 \wedge \mathbf{v}_3 \rangle = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3).
 \end{aligned}$$

应用上述等式, 有

$$(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) = (\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2).$$

性质 1.2 设 $f = f(x, y, z)$ 是 $\mathbb{R}^3$ (或者它的一个区域) 上的一个函数, 而

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$

是 $\mathbb{R}^3$ (或者它的一个区域) 上的一个向量场. 则

$$(1) \nabla \wedge (\nabla f) = \text{rot}(\text{grad} f) = \mathbf{0},$$

$$(2) \langle \nabla, \nabla \wedge \mathbf{F} \rangle = \text{div}(\text{rot} \mathbf{F}) = 0.$$

证明:

(1) 由定义, 计算得

$$\begin{aligned}
 \nabla \wedge (\nabla f) &= \nabla \wedge \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
 &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) = \mathbf{0}.
 \end{aligned}$$

(2) 根据定义, 计算得

$$\begin{aligned}
 \langle \nabla, \nabla \wedge \mathbf{F} \rangle &= \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \right\rangle \\
 &= \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right\rangle \\
 &= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0.
 \end{aligned}$$

4. 设 $\{O; e_1, e_2, e_3\}$ 是一个正交标架, σ 是 $\{1, 2, 3\}$ 的一个置换, 证明:

(1) $\{O; e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}\}$ 是一个正交标架;

(2) $\{O; e_1, e_2, e_3\}$ 与 $\{O; e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}\}$ 定向相同当且仅当 σ 是偶置换.

证明:

(1) 由于 $\{O; e_1, e_2, e_3\}$ 是正交标架, 当 $i \neq j$ 时, $\sigma(i) \neq \sigma(j)$, 故 $\langle \mathbf{e}_{\sigma(i)}, \mathbf{e}_{\sigma(j)} \rangle = 0$; 当 $i = j$ 时, $\sigma(i) = \sigma(j)$, 故 $\langle \mathbf{e}_{\sigma(i)}, \mathbf{e}_{\sigma(j)} \rangle = 1$. 因此, $\{O; e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}\}$ 是一个正交标架.

(2) $\{O; e_1, e_2, e_3\}$ 与 $\{O; e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}\}$ 定向相同 $\Leftrightarrow 1 = (e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = (-1)^{|\sigma|} \Leftrightarrow |\sigma|$ 是偶数 $\Leftrightarrow \sigma$ 是偶置换, 其中 $|\sigma|$ 是置换 σ 的长度.

5. 设 $\mathcal{T}$ 是 E^3 的一个合同变换, $\mathbf{v}$ 和 $\mathbf{w}$ 是 E^3 的两个向量, 求 $(\mathcal{T}\mathbf{v}) \wedge (\mathcal{T}\mathbf{w})$ 与 $\mathcal{T}(\mathbf{v} \wedge \mathbf{w})$ 的关系.

解: 设 $\mathcal{T}: X \mapsto XT + P$ 是 E^3 的一个合同变换, 其中 T 是一个正交矩阵.

设 $\mathbf{v} = (x^1, x^2, x^3)$, $\mathbf{w} = (y^1, y^2, y^3)$. 记 $T = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}$. 则有

$$\begin{aligned} (\mathcal{T}\mathbf{v}) \wedge (\mathcal{T}\mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (x^1 & x^2 & x^3)T \\ (y^1 & y^2 & y^3)T \end{vmatrix} = \begin{vmatrix} (\mathbf{i} & \mathbf{j} & \mathbf{k})T^{-1} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} |T| \\ &= \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} |T| = (A_1 \ A_2 \ A_3) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} |T| = (A_1 \ A_2 \ A_3) T \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} |T| \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} T |T| = |T| \mathcal{T}(\mathbf{v} \wedge \mathbf{w}). \end{aligned}$$

其中 $A_1 = x^2 y^3 - x^3 y^2$, $A_2 = -x^1 y^3 + x^3 y^1$, $A_3 = x^1 y^2 - x^2 y^1$. 因此,

$$(\mathcal{T}\mathbf{v}) \wedge (\mathcal{T}\mathbf{w}) = |T| \mathcal{T}(\mathbf{v} \wedge \mathbf{w}).$$

习题二

1. 求下列曲线的弧长与曲率:

(1) $y = ax^2$;

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

(3) $\mathbf{r}(t) = (a \cosh t, b \sinh t) (t \in \mathbb{R})$;

(4) $\mathbf{r}(t) = (t, a \cosh \frac{t}{a}) (a > 0) (t \in \mathbb{R})$.

解: (应用习题 2.)

(1) 显然, 曲线有参数表达式 $\mathbf{r}(t) = (t, at^2) (t \in \mathbb{R})$. 直接计算, 有

$$\mathbf{r}'(t) = (1, 2at), \quad \mathbf{r}''(t) = (0, 2a), \quad |\mathbf{r}'(t)| = \sqrt{1 + 4a^2t^2}.$$

因此, 弧长(作为 t 的函数) 为(注意 $a \neq 0$.)

$$s = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{1 + 4a^2u^2} du = \frac{1}{2}t\sqrt{1 + 4a^2t^2} + \frac{1}{4|a|} \log |2|a|t + \sqrt{1 + 4a^2t^2}|.$$

[上述积分的计算:

$$\text{令 } \tan \theta = 2|a|u. \text{ 则 } \sqrt{1 + 4a^2u^2} = \sec \theta. \text{ 从而, } \int \sqrt{1 + 4a^2u^2} du = \frac{1}{2|a|} \int \sec^3 \theta d\theta.$$

$$I := \int \sec^3 \theta d\theta = \int (\sec \theta \tan^2 \theta + \sec \theta) d\theta = \int \tan \theta d(\sec \theta) + \sec \theta d\theta$$

$$= \tan \theta \sec \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta = \tan \theta \sec \theta - I + \log |\sec \theta + \tan \theta|.$$

$$= \frac{1}{2}(\tan \theta \sec \theta + \log |\sec \theta + \tan \theta|) + C = \frac{1}{2}(2|a|u\sqrt{1 + 4a^2u^2} + \log |2|a|u + \sqrt{1 + 4a^2u^2}|) + C.$$

因此,

$$\int \sqrt{1 + 4a^2u^2} du = \frac{1}{2}u\sqrt{1 + 4a^2u^2} + \frac{1}{4|a|} \log |2|a|u + \sqrt{1 + 4a^2u^2}| + C.$$

]

由习题 2, 曲率

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}} = \frac{2a}{(1 + 4a^2t^2)^{\frac{3}{2}}}.$$

(2) (可以设 $a > 0$ 且 $b > 0$.) 椭圆曲线(去掉点 $(a, 0)$) 的参数表达式为

$$\mathbf{r}(t) = (a \cos t, b \sin t) (0 < t < 2\pi).$$

直接计算, 有

$$\mathbf{r}'(t) = (-a \sin t, b \cos t), \quad \mathbf{r}''(t) = (-a \cos t, -b \sin t).$$

弧长为

$$s = \int_0^t \sqrt{a^2 \sin^2 u + b^2 \cos^2 u} du = \begin{cases} at, & a = b; \\ \text{第一类椭圆积分}, & a \neq b. \end{cases}$$

曲率

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}.$$

(3) 直接计算, 有

$$\mathbf{r}'(t) = (a \sinh t, b \cosh t), \quad \mathbf{r}''(t) = (a \cosh t, b \sinh t)$$

弧长

$$s = \int_0^t \sqrt{a^2 \sinh^2 u + b^2 \cosh^2 u} du.$$

曲率

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}} = -\frac{ab}{(a^2 \sinh^2 t + b^2 \cosh^2 t)^{\frac{3}{2}}}.$$

(4) 直接计算, 有

$$\mathbf{r}'(t) = (1, \sinh \frac{t}{a}), \quad \mathbf{r}''(t) = (0, \frac{1}{a} \cosh \frac{t}{a}).$$

弧长

$$s = \int_0^t \sqrt{1 + \sinh^2 \frac{u}{a}} du = \int_0^t \cosh \frac{u}{a} du = a \sinh \frac{t}{a}.$$

曲率

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}} = \frac{\cosh \frac{t}{a}}{a(1 + \sinh^2 \frac{t}{a})^{\frac{3}{2}}}.$$

2. 设曲线 $\mathbf{r}(t) = (x(t), y(t))$, 证明它的曲率是

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

证明: 首先,

$$\mathbf{r}'(t) = (x'(t), y'(t)), \quad \mathbf{r}''(t) = (x''(t), y''(t)).$$

设 s 是曲线 $\mathbf{r}(t)$ 的弧长参数. 则 $s = s(t)$ 与 $t = t(s)$ 互为反函数.

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}, \quad \frac{dt}{ds} = \frac{1}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}},$$

$$\frac{d^2t}{ds^2} = \frac{d}{dt} \left(\frac{dt}{ds} \right) \frac{dt}{ds} = -\frac{x'(t)x''(t) + y'(t)y''(t)}{(x'(t)^2 + y'(t)^2)^2}.$$

由于平面曲线的 Frenet 标架和曲率与(同向的容许)参数选择无关, 故

$$\mathbf{t}(t) := \mathbf{t}(s(t)) = \frac{d\mathbf{r}(t)}{dt} \frac{dt}{ds} = \left(\frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right).$$

从而,

$$\begin{aligned} \dot{\mathbf{t}}(s(t)) &= \mathbf{r}''(t) \left(\frac{dt}{ds} \right)^2 + \mathbf{r}'(t) \frac{d^2t}{ds^2} \\ &= \left(-\frac{y'(t)(x'(t)y''(t) - x''(t)y'(t))}{(x'(t)^2 + y'(t)^2)^2}, \frac{x'(t)(x'(t)y''(t) - x''(t)y'(t))}{(x'(t)^2 + y'(t)^2)^2} \right) \\ \mathbf{n}(t) := \mathbf{n}(s(t)) &= \left(-\frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right). \end{aligned}$$

所以,

$$\kappa(t) = \kappa(s(t)) = \langle \dot{\mathbf{t}}(s(t)), \mathbf{n}(s(t)) \rangle = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

注: 任意参数曲线 $\mathbf{r}(t)$ 的 Frenet 标架为 $\{\mathbf{r}(t); \mathbf{t}(t), \mathbf{n}(t)\}$, 其中

$$\mathbf{t}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left(\frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right),$$

$$\mathbf{n}(t) = \frac{\mathbf{r}''(t)}{|\mathbf{r}''(t)|} = \left(-\frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right).$$

3. 设曲线 C 在极坐标 (r, θ) 下的表示为 $r = f(\theta)$, 证明 C 的曲率是

$$\kappa(\theta) = \frac{f^2(\theta) + 2\left(\frac{df}{d\theta}\right)^2 - f(\theta)\frac{d^2f}{d\theta^2}}{(f^2(\theta) + \left(\frac{df}{d\theta}\right)^2)^{\frac{3}{2}}}.$$

证明: 曲线 C 有参数表示式 $\mathbf{r}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. 直接计算, 有

$$\mathbf{r}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta),$$

$$\mathbf{r}''(\theta) = (f''(\theta)\cos\theta - 2f'(\theta)\sin\theta - f(\theta)\cos\theta, f''(\theta)\sin\theta + 2f'(\theta)\cos\theta - f(\theta)\sin\theta).$$

由习题 2, 有

$$\kappa(\theta) = \frac{x'(\theta)y''(\theta) - x''(\theta)y'(\theta)}{(x'(\theta)^2 + y'(\theta)^2)^{\frac{3}{2}}} = \frac{f^2(\theta) + 2f'(\theta)^2 - f(\theta)f''(\theta)}{(f^2(\theta) + (f'(\theta))^2)^{\frac{3}{2}}}.$$

4. 求下列曲线的曲率和挠率:

(1) $\mathbf{r}(t) = (a \cosh t, a \sinh t, bt) (a > 0);$

(2) $\mathbf{r}(t) = (3t - t^2, 3t^2, 3t + t^2);$

(3) $\mathbf{r}(t) = (a(1 - \sin t), a(1 - \cos t), bt) (a > 0);$

(4) $\mathbf{r}(t) = (at, \sqrt{2}a \log t, \frac{a}{t}) (a > 0).$

解: (应用习题 5.)

(1) 直接计算, 得

$$\mathbf{r}'(t) = (a \sinh t, a \cosh t, b),$$

$$\mathbf{r}''(t) = (a \cosh t, a \sinh t, 0),$$

$$\mathbf{r}'''(t) = (a \sinh t, a \cosh t, 0).$$

从而,

$$|\mathbf{r}'(t)| = \sqrt{a^2 \cosh 2t + b^2},$$

$$\mathbf{r}'(t) \wedge \mathbf{r}''(t) = (-ab \sinh t, ab \cosh t, -a^2),$$

$$|\mathbf{r}'(t) \wedge \mathbf{r}''(t)| = a\sqrt{b^2 \cosh 2t + a^2},$$

$$(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = \langle \mathbf{r}'(t) \wedge \mathbf{r}''(t), \mathbf{r}'''(t) \rangle = a^2 b.$$

故曲率

$$\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{a\sqrt{b^2 \cosh 2t + a^2}}{(a^2 \cosh 2t + b^2)^{\frac{3}{2}}},$$

挠率

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2} = \frac{b}{b^2 \cosh 2t + a^2}.$$

(2) 直接计算, 得

$$\mathbf{r}'(t) = (3 - 2t, 6t, 3 + 2t),$$

$$\mathbf{r}''(t) = (-2, 6, 2),$$

$$\mathbf{r}'''(t) = \mathbf{0}.$$

从而,

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{2(4t^2 + 18t + 9)}, \\ \mathbf{r}'(t) \wedge \mathbf{r}''(t) &= (-18, -12, 18), \\ |\mathbf{r}'(t) \wedge \mathbf{r}''(t)| &= 6\sqrt{22}, \\ (\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) &= 0. \end{aligned}$$

故曲率

$$\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{3\sqrt{11}}{(4t^2 + 18t + 9)^{\frac{3}{2}}},$$

挠率

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2} = 0.$$

(3) 直接计算, 得

$$\begin{aligned} \mathbf{r}'(t) &= (-a \cos t, a \sin t, b), \\ \mathbf{r}''(t) &= (a \sin t, a \cos t, 0), \\ \mathbf{r}'''(t) &= (a \cos t, -a \sin t, 0). \end{aligned}$$

从而,

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{a^2 + b^2}, \\ \mathbf{r}'(t) \wedge \mathbf{r}''(t) &= (-ab \cos t, ab \sin t, -a^2), \\ |\mathbf{r}'(t) \wedge \mathbf{r}''(t)| &= a\sqrt{a^2 + b^2}, \\ (\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) &= \langle \mathbf{r}'(t) \wedge \mathbf{r}''(t), \mathbf{r}'''(t) \rangle = -a^2b. \end{aligned}$$

故曲率

$$\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{\frac{3}{2}}} = \frac{a}{a^2 + b^2},$$

挠率

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2} = \frac{-a^2b}{a^2(a^2 + b^2)} = -\frac{b}{a^2 + b^2}.$$

(4) (注意到 $t \in (0, +\infty)$). 直接计算, 得

$$\begin{aligned} \mathbf{r}'(t) &= (a, \frac{\sqrt{2}a}{t}, -\frac{a}{t^2}), \\ \mathbf{r}''(t) &= (0, -\frac{\sqrt{2}a}{t^2}, \frac{2a}{t^3}), \\ \mathbf{r}'''(t) &= (0, \frac{2\sqrt{2}a}{t^3}, -\frac{6a}{t^4}). \end{aligned}$$

从而,

$$\begin{aligned} |\mathbf{r}'(t)| &= \frac{a(t^2 + 1)}{t^2}, \\ \mathbf{r}'(t) \wedge \mathbf{r}''(t) &= (\frac{\sqrt{2}a^2}{t^4}, -\frac{2a^2}{t^3}, -\frac{\sqrt{2}a^2}{t^2}), \\ |\mathbf{r}'(t) \wedge \mathbf{r}''(t)| &= \frac{\sqrt{2}a^2(t^2 + 1)}{t^4}, \end{aligned}$$

$$(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = \langle \mathbf{r}'(t) \wedge \mathbf{r}''(t), \mathbf{r}'''(t) \rangle = \frac{2\sqrt{2}a^3}{t^6}.$$

故曲率

$$\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2}t^2}{a(t^2 + 1)^2},$$

挠率

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2} = \frac{\sqrt{2}t^2}{a(t^2 + 1)^2}.$$

5. 证明: E^3 中的正则曲线 $\mathbf{r}(t)$ 的曲率和挠率分别是

$$\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3},$$

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2}.$$

证明: 设 s 是曲线 $\mathbf{r}(t)$ 的弧长参数. 则 $s = s(t)$ 与 $t = t(s)$ 互为反函数. 由于空间曲线的 Frenet 标架和曲率与(容许的)参数选取无关, 故

$$\begin{aligned} \mathbf{t}(t) &:= \mathbf{t}(s(t)) = \frac{d\mathbf{r}(t(s))}{ds} = \mathbf{r}'(t) \frac{dt}{ds}, \quad \frac{dt}{ds} = \frac{1}{|\mathbf{r}'(t)|}. \\ \dot{\mathbf{t}}(s(t)) &= \mathbf{r}''(t) \left(\frac{dt}{ds}\right)^2 + \mathbf{r}'(t) \frac{d^2t}{ds^2}, \quad \mathbf{n}(s(t)) = \frac{1}{\kappa(s(t))} \dot{\mathbf{t}}(s(t)), \\ \ddot{\mathbf{t}}(s(t)) &= \mathbf{r}'''(t) \left(\frac{dt}{ds}\right)^3 + 3\mathbf{r}''(t) \frac{dt}{ds} \frac{d^2t}{ds^2} + \mathbf{r}'(t) \frac{d^3t}{ds^3}, \\ \mathbf{b}(s(t)) &= \mathbf{t}(s(t)) \wedge \mathbf{n}(s(t)) = \frac{1}{\kappa(s(t))} \mathbf{t}(s(t)) \wedge \dot{\mathbf{t}}(s(t)) \\ &= \frac{1}{\kappa(s(t))} \mathbf{r}'(t) \frac{dt}{ds} \wedge \left(\mathbf{r}''(t) \left(\frac{dt}{ds}\right)^2 + \mathbf{r}'(t) \frac{d^2t}{ds^2} \right) \\ &= \frac{1}{\kappa(s(t))} \left(\frac{dt}{ds}\right)^3 \mathbf{r}'(t) \wedge \mathbf{r}''(t) \end{aligned}$$

从而, 曲率

$$\kappa(t) = \kappa(s(t)) = \left(\frac{dt}{ds}\right)^3 |\mathbf{r}'(t) \wedge \mathbf{r}''(t)| = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

由于 $\dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s)$, 故

$$\begin{aligned} \ddot{\mathbf{t}}(s) &= \dot{\kappa}(s)\mathbf{n}(s) + \kappa(s)\dot{\mathbf{n}}(s) = \dot{\kappa}(s)\mathbf{n}(s) + \kappa(s)(-\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s)) \\ &= -\kappa(s)^2\mathbf{t}(s) + \dot{\kappa}(s)\mathbf{n}(s) + \kappa(s)\tau(s)\mathbf{b}(s). \end{aligned}$$

从而,

$$\begin{aligned} \kappa(s(t))\tau(s(t)) &= \langle \ddot{\mathbf{t}}(s(t)), \mathbf{b}(s(t)) \rangle \\ &= \langle \mathbf{r}'''(t) \left(\frac{dt}{ds}\right)^3 + 3\mathbf{r}''(t) \frac{dt}{ds} \frac{d^2t}{ds^2} + \mathbf{r}'(t) \frac{d^3t}{ds^3}, \frac{1}{\kappa(s(t))} \mathbf{t}(s(t)) \wedge \dot{\mathbf{t}}(s(t)) \rangle \\ &= \frac{1}{\kappa(s(t))} \left(\frac{dt}{ds}\right)^6 (\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)). \end{aligned}$$

因此,

$$\tau(t) = \tau(s(t)) = \frac{1}{\kappa(t)^2} \frac{1}{|\mathbf{r}'(t)|^6} (\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2}.$$

注: 任意参数曲线 $\mathbf{r}(t)$ 的 Frenet 标架为 $\{\mathbf{r}(t); \mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$, 其中

$$\begin{aligned} \mathbf{t}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left(\frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}, \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}}, \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} \right), \\ \mathbf{n}(t) &= \frac{\mathbf{r}''(t)}{|\mathbf{r}''(t)|}, \\ \mathbf{b}(t) &= \frac{\mathbf{r}'(t) \wedge \mathbf{r}''(t)}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}. \end{aligned}$$

6. 证明: 曲线

$$\mathbf{r}(s) = \left(\frac{(1+s)^{\frac{3}{2}}}{3}, \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{s}{\sqrt{2}} \right) \quad (-1 < s < 1)$$

以 s 为弧长参数, 并求它的曲率、挠率和 Frenet 标架.

证明: 由于

$$\mathbf{r}'(s) = \left(\frac{(1+s)^{\frac{1}{2}}}{2}, -\frac{(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}} \right) \quad (-1 < s < 1),$$

$|\mathbf{r}'(s)| = \sqrt{\frac{1+s}{4} + \frac{1-s}{4} + \frac{1}{2}} = 1$. 故 s 是弧长参数. 从而,

$$\mathbf{t}(s) = \mathbf{r}'(s),$$

$$\dot{\mathbf{t}}(s) = \left(\frac{(1+s)^{-\frac{1}{2}}}{4}, -\frac{(1-s)^{-\frac{1}{2}}}{4}, 0 \right).$$

故曲率 $\kappa(s) = |\dot{\mathbf{t}}(s)| = \sqrt{\frac{1}{8(1-s^2)}} = \frac{\sqrt{2}(1-s^2)^{-\frac{1}{2}}}{4}$. 从而,

$$\mathbf{n}(s) = \frac{1}{\kappa} \dot{\mathbf{t}}(s) = \left(\frac{\sqrt{2}(1-s)^{\frac{1}{2}}}{2}, \frac{\sqrt{2}(1+s)^{\frac{1}{2}}}{2}, 0 \right),$$

$$\dot{\mathbf{n}}(s) = \left(-\frac{\sqrt{2}(1-s)^{-\frac{1}{2}}}{4}, \frac{\sqrt{2}(1+s)^{-\frac{1}{2}}}{4}, 0 \right),$$

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s) = \left(-\frac{(1+s)^{\frac{1}{2}}}{2}, \frac{(1-s)^{\frac{1}{2}}}{2}, \frac{\sqrt{2}}{2} \right).$$

挠率

$$\tau(s) = \langle \dot{\mathbf{n}}(s), \mathbf{b}(s) \rangle (= -\langle \dot{\mathbf{b}}(s), \mathbf{n}(s) \rangle) = \frac{\sqrt{2}(1-s^2)^{-\frac{1}{2}}}{4}.$$

最后, 曲线 $\mathbf{r}(s)$ 的 Frenet 标架为 $\{\mathbf{r}(s); \mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$, 其中 $\mathbf{t}, \mathbf{n}, \mathbf{b}$ 如上.

7. 设曲线

$$\mathbf{r}(t) = \begin{cases} (e^{-\frac{1}{t^2}}, t, 0), & t < 0; \\ (0, 0, 0), & t = 0; \\ (0, t, e^{-\frac{1}{t^2}}), & t > 0. \end{cases} \quad (1)$$

(1) 证明: $\mathbf{r}(t)$ 是一条正则曲线, 且在 $t=0$ 处曲率 $\kappa=0$;

(2) 求 $\mathbf{r}(t) (t \neq 0)$ 时的 Frenet 标架, 并讨论 $t \rightarrow 0$ 时, Frenet 标架的极限.

解: (1) 首先注意到对任意正整数 k , $(e^{-\frac{1}{t^2}})^{(k)} = \frac{P_k(t)}{t^{n_k}} e^{-\frac{1}{t^2}}$, 其中 n_k 是正整数, $P_k(t)$ 是关于 t 的一个多项式. 这由归纳法容易得到.

当 $t < 0$ 时,

$$\mathbf{r}'(t) = \left(\frac{2}{t^3} e^{-\frac{1}{t^2}}, 1, 0\right), \quad |\mathbf{r}'(t)| = \left(1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}\right)^{\frac{1}{2}},$$

$$\mathbf{r}''(t) = \left(\frac{-6t^2 + 4}{t^6} e^{-\frac{1}{t^2}}, 0, 0\right), \quad \mathbf{r}'(t) \wedge \mathbf{r}''(t) = \left(0, 0, \frac{6t^2 - 4}{t^6} e^{-\frac{1}{t^2}}\right),$$

$$\mathbf{r}^{(k)}(t) = \left(\frac{P_k(t)}{t^{n_k}} e^{-\frac{1}{t^2}}, 0, 0\right) (k \geq 2).$$

由上知, $\mathbf{r}(t) (t < 0)$ 是光滑的且 $\mathbf{r}'(t) \neq \mathbf{0}$, 即是正则的.

曲线在 $t=0$ 处的第 k -阶左导数为 ($k \geq 0$)

$$\mathbf{r}^{(k)}(0^-) = \lim_{t \rightarrow 0^-} \mathbf{r}^{(k)}(t) = \begin{cases} \mathbf{0} & , \quad k \neq 1; \\ (0, 1, 0), & k = 1. \end{cases} \quad (2)$$

类似地, 当 $t > 0$ 时,

$$\mathbf{r}'(t) = \left(0, 1, \frac{2}{t^3} e^{-\frac{1}{t^2}}\right), \quad |\mathbf{r}'(t)| = \left(1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}\right)^{\frac{1}{2}},$$

$$\mathbf{r}''(t) = \left(0, 0, \frac{-6t^2 + 4}{t^6} e^{-\frac{1}{t^2}}\right), \quad \mathbf{r}'(t) \wedge \mathbf{r}''(t) = \left(\frac{-6t^2 + 4}{t^6} e^{-\frac{1}{t^2}}, 0, 0\right),$$

$$\mathbf{r}^{(k)}(t) = \left(0, 0, \frac{P_k(t)}{t^{n_k}} e^{-\frac{1}{t^2}}\right) (k \geq 2).$$

故 $\mathbf{r}(t) (t > 0)$ 是正则的.

曲线在 $t=0$ 处的第 k -阶右导数为 ($k \geq 0$, $\mathbf{r}^{(0)} = \mathbf{r}$)

$$\mathbf{r}^{(k)}(0^+) = \lim_{t \rightarrow 0^+} \mathbf{r}^{(k)}(t) = \begin{cases} \mathbf{0} & , \quad k \neq 1; \\ (0, 1, 0), & k = 1. \end{cases} \quad (3)$$

$$= \mathbf{r}^{(k)}(0^-).$$

因此, 曲线在 $t=0$ 处也是正则的. 综上, 曲线 $\mathbf{r}(t) (t \in \mathbb{R})$ 是正则的. 从而, 其曲率(定义在 $\mathbb{R}$ 上)是光滑的.

由习题 5, 曲线 $\mathbf{r}(t) (t \neq 0)$ 的曲率为

$$\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|6t^2 - 4| e^{-\frac{1}{t^2}}}{t^6 \left(1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}\right)^{\frac{3}{2}}}.$$

而 $\kappa(t)$ 在 $\mathbb{R}$ 上是光滑的; 特别地, 在 $t=0$ 处是连续的. 故有

$$\kappa(0) = \lim_{t \rightarrow 0} \kappa(t) = \lim_{t \rightarrow 0} \frac{|6t^2 - 4| e^{-\frac{1}{t^2}}}{t^6 \left(1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}\right)^{\frac{3}{2}}} = 0.$$

(2) 由习题 5 后面的注, 曲线 $\mathbf{r}(t) (t \neq 0)$ 的 Frenet 标架为 $\{\mathbf{r}(t); \mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$, 其中

$$\mathbf{t}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{cases} \left(\frac{\frac{2}{t^3}e^{-\frac{1}{t^2}}}{(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}}, \frac{1}{(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}}, 0 \right), & t < 0; \\ \left(0, \frac{1}{(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}}, \frac{\frac{2}{t^3}e^{-\frac{1}{t^2}}}{(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}} \right), & t > 0. \end{cases} \quad (4)$$

$$\mathbf{n}(t) = \begin{cases} \left(\frac{\operatorname{sgn}(-6t^2 + 4)}{(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}}, \frac{2\operatorname{sgn}(6t^2 - 4)e^{-\frac{1}{t^2}}}{t^3(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}}, 0 \right), & t < 0; \\ \left(0, \frac{2\operatorname{sgn}(6t^2 - 4)e^{-\frac{1}{t^2}}}{t^3(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}}, \frac{\operatorname{sgn}(-6t^2 + 4)}{(1 + \frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{1}{2}}} \right), & t > 0. \end{cases} \quad (5)$$

$$\mathbf{b}(t) = \mathbf{t}(t) \wedge \mathbf{n}(t) = \begin{cases} (0, 0, -1), & t < 0; \\ (1, 0, 0), & t > 0. \end{cases} \quad (6)$$

故当 $t \rightarrow 0^-$ 时, 曲线的 Frenet 标架为 $\{\mathbf{r}(0^-) = \mathbf{0}; \mathbf{t}(0^-), \mathbf{n}(0^-), \mathbf{b}(0^-)\}$, 其中

$$\mathbf{t}(0^-) = (0, 1, 0), \quad \mathbf{n}(0^-) = (1, 0, 0), \quad \mathbf{b}(0^-) = (0, 0, -1).$$

而当 $t \rightarrow 0^+$ 时, 曲线的 Frenet 标架为 $\{\mathbf{r}(0^+) = \mathbf{0}; \mathbf{t}(0^+), \mathbf{n}(0^+), \mathbf{b}(0^+)\}$, 其中

$$\mathbf{t}(0^+) = (0, 1, 0), \quad \mathbf{n}(0^+) = (0, 0, 1), \quad \mathbf{b}(0^+) = (1, 0, 0).$$

因此, 当 $t \rightarrow 0$ 时, 曲线的 Frenet 标架的极限不存在(左、右极限存在但不相等).

注: 从上面可以看到, 正则曲线在曲率为 0 的点处, Frenet 标架的极限未必存在.

8. 设平面正则曲线 $C: \mathbf{r} = \mathbf{r}(t)$ 不过点 P_0 , 而 $\mathbf{r}(t_0)$ 是 C 与 P_0 距离最近的点, 证明: 向量 $\mathbf{r}(t_0) - \overrightarrow{OP_0}$ 与 $\mathbf{r}'(t_0)$ 垂直.

证明: 设 $\mathbf{p}_0$ 是 P_0 的位置向量. 由假设, $\mathbf{r}(t_0)$ 是距离函数 $d(\mathbf{r}(t), \mathbf{p}_0)$ 的最小值点, 也是极小值点. 从而, $\mathbf{r}(t_0)$ 是 $d^2(\mathbf{r}(t), \mathbf{p}_0) = \langle \mathbf{r}(t) - \mathbf{p}_0, \mathbf{r}(t) - \mathbf{p}_0 \rangle$ 的极小值点. 故 $d^2(\mathbf{r}(t), \mathbf{p}_0)$ 在此点的导数为 0, 即: $(d^2(\mathbf{r}(t), \mathbf{p}_0))'(t_0) = 2\langle \mathbf{r}'(t_0), \mathbf{r}(t_0) - \mathbf{p}_0 \rangle = 0$. 从而, $\mathbf{r}(t_0) - \mathbf{p}_0 \perp \mathbf{r}'(t_0)$.

9. (1) 设 E^3 中曲线 C 的所有切线过一个定点, 证明 C 是直线.

(2) 证明: 所有主法线过定点的曲线是圆.

证明: (1) 设 P_0 是弧长参数曲线 $C: \mathbf{r} = \mathbf{r}(s)$ 的切线过的定点, 其位置向量为 $\mathbf{p}_0$.

方法一:

由假设,

$$\mathbf{r}(s) - \mathbf{p}_0 = \lambda(s)\mathbf{t}(s),$$

其中 $\lambda(s) = \langle \mathbf{r}(s) - \mathbf{p}_0, \mathbf{t}(s) \rangle$ 是一个光滑函数. 对上式两边求导, 有

$$\mathbf{t}(s) = \lambda(s)'\mathbf{t}(s) + \lambda(s)\dot{\mathbf{t}}(s) = \lambda(s)'\mathbf{t}(s) + \lambda(s)\kappa(s)\mathbf{n}(s).$$

由于 $\mathbf{t}(s), \mathbf{n}(s)$ 处处线性无关, 有

$$\lambda(s)' = 1, \quad \lambda(s)\kappa(s) = 0.$$

从而, $\lambda(s)$ 不恒为 0. 因此, 由 $\kappa(s)$ 的连续性, 知 $\kappa(s) \equiv 0$. 故, C 是直线.

方法二:

由假设, $\mathbf{r}(s) - \mathbf{p}_0 // \mathbf{t}(s) \Leftrightarrow (\mathbf{r}(s) - \mathbf{p}_0) \wedge \mathbf{t}(s) = 0$. 若 $\mathbf{p}_0$ 不在 C 上, 则由习题一第 2 题 (2), C 是一条过 P_0 的直线, 矛盾. 因此, P_0 在 C 上. 设 $\mathbf{r}(s_0) = \mathbf{p}_0$. 由习题一第 2 题 (2), $\mathbf{r}(s)(s > s_0)$ 和 $\mathbf{r}(s)(s < s_0)$ 是两条射线. 而 $\mathbf{r}(s)$ 在 $s = s_0$ 处可微, 即: 只有一条切线, 故 C 必然是一条直线.

(2) 设弧长参数曲线 $C: \mathbf{r} = \mathbf{r}(s)$ 的主法线过定点 P_0 (其位置向量为 $\mathbf{p}_0$). 则有(注意: 假设了主法线存在, 故曲率恒不为 0)

$$\mathbf{r}(s) - \mathbf{p}_0 = \lambda(s)\mathbf{n}(s),$$

其中 $\lambda(s) = \langle \mathbf{r}(s) - \mathbf{p}_0, \mathbf{n}(s) \rangle$ 是一个光滑函数. 对上式两边求导并应用 Frenet 公式, 有

$$\mathbf{t}(s) = \lambda(s)'\mathbf{n}(s) - \lambda(s)\kappa(s)\mathbf{t}(s) + \lambda(s)\tau(s)\mathbf{b}(s).$$

由于 $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ 处处线性无关, 有

$$1 - \lambda(s)\kappa(s) = 0, \quad \lambda(s)' = 0, \quad \lambda(s)\tau(s) = 0.$$

由 $\lambda(s)' = 0$, 知 $\lambda(s) = \lambda$ 是常数. 而由 $1 - \lambda(s)\kappa(s) = 0$, 有 $\lambda \neq 0$ 且 $\kappa(s) = \frac{1}{\lambda}$ 是常数. 而 $\lambda(s)\tau(s) = 0$, 故 $\tau(s) \equiv 0$. 因此, 由定理 3.1 (p.22), C 是平面曲线. 而由例 2.2 (p.17), 知 C 是圆.

10. 设 $\mathcal{T}(X) = X\mathbf{T} + P$ 是 E^3 的一个合同变换, $\det \mathbf{T} = -1$. $\mathbf{r}(t)$ 是 E^3 的正规曲线. 求曲线 $\tilde{\mathbf{r}}(t) = \mathcal{T} \circ \mathbf{r}(t)$ 与曲线 $\mathbf{r}(t)$ 的弧长参数、曲率、挠率间的关系.

证明: 设 $\tilde{s}$ 是 $\tilde{\mathbf{r}}(t)$ 的弧长参数. 由于 $\mathcal{T}$ 保持距离并且 $\tilde{\mathbf{r}}'(t) = \mathbf{r}'(t)\mathbf{T}$, 有

$$\frac{d\tilde{s}}{dt} = |\tilde{\mathbf{r}}'(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}.$$

故 $\tilde{s} - s = \text{常数}$, 即: $\mathbf{r}(t)$ 与 $\tilde{\mathbf{r}}(t)$ 的弧长参数相同. 因此, 下面将 s 看作 $\tilde{\mathbf{r}}(t)$ 的弧长参数.

弧长参数曲线 $\tilde{\mathbf{r}}(s)$ 的 Frenet 标架为

$$\tilde{\mathbf{t}}(s) = \frac{d\tilde{\mathbf{r}}(s)}{ds} = \frac{d\mathbf{r}(s)}{ds}\mathbf{T} = \mathbf{t}\mathbf{T},$$

$$\frac{d\tilde{\mathbf{t}}(s)}{ds} = \frac{d\mathbf{t}}{ds}\mathbf{T},$$

因此, $\tilde{\mathbf{n}}(s) = \mathbf{n}(s)\mathbf{T}$, $\tilde{\kappa}(s) = \kappa(s)$. 由于 $\det \mathbf{T} = -1$, 有

$$\tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}}(s) \wedge \tilde{\mathbf{n}}(s) = (\mathbf{t}(s)\mathbf{T}) \wedge (\mathbf{n}(s)\mathbf{T}) = \det \mathbf{T}(\mathbf{t}(s) \wedge \mathbf{n}(s))\mathbf{T} = -\mathbf{b}(s)\mathbf{T}.$$

其中倒数第二个等式应用了习题一, 5 (p.13). 故

$$\tilde{\tau}(s) = -\langle \dot{\tilde{\mathbf{b}}}(s), \tilde{\mathbf{n}}(s) \rangle = -\langle -\dot{\mathbf{b}}(s)\mathbf{T}, \mathbf{n}(s)\mathbf{T} \rangle = \langle \dot{\mathbf{b}}(s), \mathbf{n}(s) \rangle = -\tau(s).$$

换回参数 t , 有 $\tilde{\kappa}(t) = \kappa(t)$, $\tilde{\tau}(t) = -\tau(t)$.

11. 设弧长参数曲线 $\mathbf{r}(s)$ 的曲线 $\kappa > 0$, 挠率 $\tau > 0$, $\mathbf{b}(s)$ 是 C 的副法线向量, 定义曲线 $\tilde{C}$:

$$\tilde{\mathbf{r}}(s) = \int_0^s \mathbf{b}(u)du.$$

(1) 证明 s 是曲线 $\tilde{C}$ 的弧长参数且 $\tilde{\kappa} = \tau$, $\tilde{\tau} = \kappa$;

(2) 求 $\tilde{C}$ 的 Frenet 标架.

解: 由于 $|\tilde{\mathbf{r}}'(s)| = |\mathbf{b}(s)| = 1$, 故 s 是曲线 $\tilde{C}: \tilde{\mathbf{r}}(s)$ 的弧长参数. 设曲线 $\tilde{C}$ 的 Frenet 标架为 $\{\tilde{\mathbf{r}}(s), \tilde{\mathbf{t}}(s), \tilde{\mathbf{n}}(s), \tilde{\mathbf{b}}(s)\}$. 首先, 有 $\tilde{\mathbf{t}}(s) = \mathbf{b}(s)$. 由定义及 Frenet 公式, 曲线 $\tilde{C}$ 的曲率

$$\tilde{\kappa}(s) = |\dot{\tilde{\mathbf{t}}}(s)| = |\dot{\mathbf{b}}(s)| = |-\tau(s)\mathbf{n}| = \tau(s).$$

从而, $\tilde{\mathbf{n}}(s) = -\mathbf{n}(s)$ 且 $\dot{\tilde{\mathbf{n}}}(s) = -\dot{\mathbf{n}}(s) = \kappa(s)\mathbf{t} - \tau(s)\mathbf{b}(s)$. 因此,

$$\tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}}(s) \wedge \tilde{\mathbf{n}}(s) = -\mathbf{b}(s) \wedge \mathbf{n}(s) = \mathbf{t}(s).$$

故 $\tilde{C}$ 上每点的 Frenet 标架为 $\{\tilde{\mathbf{r}}(s), \mathbf{b}(s), -\mathbf{n}(s), \mathbf{t}(s)\}$. 最后, 曲线 $\tilde{C}$ 的挠率

$$\tilde{\tau}(s) = \langle \dot{\tilde{\mathbf{n}}}(s), \tilde{\mathbf{b}}(s) \rangle = \langle \kappa(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s), \mathbf{t}(s) \rangle = \kappa(s).$$

12. 给定弧长参数曲线 $\mathbf{r}(s)$, 它的曲率和挠率分别是 $\kappa = \kappa(s), \tau = \tau(s)$; $\mathbf{r}(s)$ 的单位切向量 $\mathbf{t}(s)$ 可看作单位球面 S^2 上的一条曲线, 称为曲线 $\mathbf{r}(s)$ 的切线像. 证明: 曲线 $\tilde{\mathbf{r}}(s) := \mathbf{t}(s)$ 的曲率、挠率分别是

$$\tilde{\kappa}(s) = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2},$$

$$\tilde{\tau}(s) = \frac{\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)}{\kappa\left(1 + \left(\frac{\tau}{\kappa}\right)^2\right)}.$$

证明: 由 Frenet 公式, $\tilde{\mathbf{r}}'(s) = \dot{\mathbf{t}}(s) = \kappa\mathbf{n}$. (故 $|\tilde{\mathbf{r}}'(s)| = \kappa$, s 未必是曲线 $\tilde{\mathbf{r}}(s)$ 弧长参数.) 将应用习题 5 计算曲线 $\tilde{\mathbf{r}}(s)$ 的曲率和挠率. 根据 Frenet 公式,

$$\tilde{\mathbf{r}}''(s) = \dot{\kappa}\mathbf{n} + \kappa\dot{\mathbf{n}} = -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \kappa\tau\mathbf{b},$$

$$\tilde{\mathbf{r}}'''(s) = -3\kappa\dot{\kappa}\mathbf{t} + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\dot{\kappa}\tau + \kappa\dot{\tau})\mathbf{b}.$$

从而,

$$\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s) = \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b}, \quad |\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)| = \kappa^2\sqrt{\kappa^2 + \tau^2},$$

由习题 5, 有

$$\tilde{\kappa}(s) = \frac{|\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)|}{|\tilde{\mathbf{r}}'(s)|^3} = \frac{\kappa^2\sqrt{\kappa^2 + \tau^2}}{\kappa^3} = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}.$$

由于

$$(\tilde{\mathbf{r}}'(s), \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s)) = \kappa^3(2\dot{\kappa}\tau + \kappa\dot{\tau}) - 3\kappa^3\dot{\kappa}\tau = \kappa^3(\dot{\kappa}\tau - \kappa\dot{\tau}),$$

故有

$$\tilde{\tau}(s) = \frac{(\tilde{\mathbf{r}}'(s), \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s))}{|\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)|^2} = \frac{\kappa^3(\dot{\kappa}\tau - \kappa\dot{\tau})}{\kappa^4(\kappa^2 + \tau^2)} = \frac{\frac{\dot{\kappa}\tau - \kappa\dot{\tau}}{\kappa^2}}{\kappa + \frac{\tau^2}{\kappa}} = \frac{\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)}{\kappa\left(1 + \left(\frac{\tau}{\kappa}\right)^2\right)}.$$

13. (1) 求曲率 $\kappa(s) = \frac{a}{a^2+s^2}$ (s 为弧长参数) 的平面曲线;

(2) 求曲率 $\kappa(s) = \frac{1}{\sqrt{a^2-s^2}}$ (s 为弧长参数) 的平面曲线.

解: 一般地, 给定曲率 $\kappa(s)$ (以 s 为弧长参数), 求解平面曲线的步骤如下:

设 $\mathbf{t}(s) = \frac{d\mathbf{r}(s)}{ds} = \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (\cos\theta(s), \sin\theta(s))$ (根据引理 1.2 (p.156), 总存在这样的 $\theta(s)$ 且它至少是连续可微的), 则 $\kappa(s) = \langle \dot{\mathbf{t}}(s), \mathbf{n}(s) \rangle = \frac{d\theta}{ds}$. 故

$$\theta(s) = \int_0^s \kappa(t)dt + c,$$

其中 c 为常数. 从而,

$$\begin{aligned}\frac{dx(s)}{ds} &= \cos\left(\int_0^s \kappa(t)dt + c\right), \\ \frac{dy(s)}{ds} &= \sin\left(\int_0^s \kappa(t)dt + c\right),\end{aligned}$$

因此,

$$\begin{aligned}x(s) &= \int_0^s \cos\left(\int_0^u \kappa(t)dt\right)du + c + c_1, \\ y(s) &= \int_0^s \sin\left(\int_0^u \kappa(t)dt\right)du + c + c_2,\end{aligned}$$

其中 c_1, c_2 是常数. 要求的平面曲线为

$$\mathbf{r}(s) = \left(\int_0^s \cos\left(\int_0^u \kappa(t)dt\right)du + c\right) + c_1, \int_0^s \sin\left(\int_0^u \kappa(t)dt\right)du + c + c_2$$

[或者, 取 $c = c_1 = c_2 = 0$, 先得到一条曲线

$$\mathbf{r}(s) = \left(\int_0^s \cos\left(\int_0^u \kappa(t)dt\right)du\right), \int_0^s \sin\left(\int_0^u \kappa(t)dt\right)du).$$

然后, 根据平面曲线论基本定理 4.4, 通过平面刚体运动得到所有满足条件的曲线.]

代入上述公式, 得要求的平面曲线($a > 0$):

(1) $\mathbf{r}(s) = (a \log(s + \sqrt{a^2 + s^2}), \sqrt{a^2 + s^2})$ (及其与任意平面刚体运动的合成);

(2) $\mathbf{r}(s) = (\frac{1}{2a}s\sqrt{a^2 - s^2} + \frac{a}{2} \arcsin \frac{s}{a}, \frac{s^2}{2a})$ (及其与任意平面刚体运动的合成).

14. 证明: 对 E^3 的弧长参数曲线 $\mathbf{r}(s)$, 有

(1) $(\frac{d\mathbf{r}}{ds}, \frac{d^2\mathbf{r}}{ds^2}, \frac{d^3\mathbf{r}}{ds^3}) = \kappa^2 \boldsymbol{\tau}$;

(2) $(\frac{d\mathbf{t}}{ds}, \frac{d^2\mathbf{t}}{ds^2}, \frac{d^3\mathbf{t}}{ds^3}) = \kappa^3(\kappa\dot{\tau} - \dot{\kappa}\tau) = \kappa^5 \frac{d}{ds}(\frac{\tau}{\kappa})$.

证明: (1) 应用 Frenet 公式, 有 $\frac{d\mathbf{r}}{ds} = \mathbf{t}$, $\frac{d^2\mathbf{r}}{ds^2} = \dot{\mathbf{t}} = \kappa\mathbf{n}$, $\frac{d^3\mathbf{r}}{ds^3} = \ddot{\mathbf{t}} = \dot{\kappa}\mathbf{n} + \kappa\dot{\mathbf{n}} = -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \kappa\tau\mathbf{b}$, $\frac{d\mathbf{r}}{ds} \wedge \frac{d^2\mathbf{r}}{ds^2} = \kappa\mathbf{b}$. 故

$$\left(\frac{d\mathbf{r}}{ds}, \frac{d^2\mathbf{r}}{ds^2}, \frac{d^3\mathbf{r}}{ds^3}\right) = \left\langle \frac{d\mathbf{r}}{ds} \wedge \frac{d^2\mathbf{r}}{ds^2}, \frac{d^3\mathbf{r}}{ds^3} \right\rangle = \langle \kappa\mathbf{b}, -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \kappa\tau\mathbf{b} \rangle = \kappa^2\tau.$$

(2) 由 Frenet 公式, $\frac{d^3\mathbf{t}}{ds^3} = -3\kappa\dot{\kappa}\mathbf{t} + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\dot{\kappa}\tau + \kappa\dot{\tau})\mathbf{b}$. 而 $\frac{d\mathbf{t}}{ds} \wedge \frac{d^2\mathbf{t}}{ds^2} = \frac{d^2\mathbf{r}}{ds^2} \wedge \frac{d^3\mathbf{r}}{ds^3} = \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b}$, 故

$$\begin{aligned}\left(\frac{d\mathbf{t}}{ds}, \frac{d^2\mathbf{t}}{ds^2}, \frac{d^3\mathbf{t}}{ds^3}\right) &= \left\langle \frac{d\mathbf{t}}{ds} \wedge \frac{d^2\mathbf{t}}{ds^2}, \frac{d^3\mathbf{t}}{ds^3} \right\rangle = \langle \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b}, -3\kappa\dot{\kappa}\mathbf{t} + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\dot{\kappa}\tau + \kappa\dot{\tau})\mathbf{b} \rangle \\ &= -3\kappa^3\dot{\kappa}\tau + \kappa^3(2\kappa\dot{\tau} + \kappa\dot{\tau}) = \kappa^3(\kappa\dot{\tau} - \dot{\kappa}\tau) = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right).\end{aligned}$$

15. 证明: 满足条件

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right)^2 = \text{常数}$$

的弧长参数曲线, 或者是球面曲线, 或者 κ 是常数.

证明: 假设弧长参数曲线 $\mathbf{r}(s)$ 的曲率函数 κ 不是常数. 考虑向量场

$$\mathbf{p}(s) = \mathbf{r}(s) + \frac{1}{\kappa}\mathbf{n} + \frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\mathbf{b}.$$

求导数, 有

$$\begin{aligned}\mathbf{p}'(s) &= \mathbf{t} + \frac{1}{\kappa} \dot{\mathbf{n}} + \left(-\frac{\dot{\kappa}}{\kappa^2}\right) \mathbf{n} + \frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right) \dot{\mathbf{b}} + \frac{d}{ds} \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right) \mathbf{b} \\ &= \mathbf{t} - \mathbf{t} + \frac{\tau}{\kappa} \mathbf{n} - \frac{\dot{\kappa}}{\kappa^2} \mathbf{n} + \frac{\dot{\kappa}}{\kappa^2} \mathbf{n} + \frac{d}{ds} \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right) \mathbf{b} = \left(\frac{\tau}{\kappa} + \frac{d}{ds} \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right)\right) \mathbf{b}.\end{aligned}$$

而已知 $\left(\frac{1}{\kappa}\right)^2 + \left[\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right]^2 = \text{常数}$, 求导数, 得

$$-2 \frac{\dot{\kappa}}{\kappa^3} - 2 \frac{\dot{\kappa}}{\kappa^2 \tau} \frac{d}{ds} \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right) = 0.$$

由于 κ 不是常数, 故 $\dot{\kappa}$ 不恒为 0. 由连续性, 得到

$$\frac{\tau}{\kappa} + \frac{d}{ds} \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right) \equiv 0.$$

因此, $\mathbf{p}'(s) = 0$. 故 $\mathbf{p}(s)$ 是常向量, 记为 $\mathbf{p}_0$. 而

$$|\mathbf{r}(s) - \mathbf{p}_0|^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right)^2 = \text{常数},$$

故曲线 $\mathbf{r}(s)$ 在一个球面上.

注: (1) 设弧长参数曲线 $\mathbf{r}(s)$ 在一个半径为 a 的球面上, 即: $|\mathbf{r}(s) - \mathbf{p}_0|^2 = a^2$ 对某个常向量 $\mathbf{p}_0$. 求导数, 有 $2\langle \mathbf{r}(s) - \mathbf{p}_0, \mathbf{t} \rangle = 0$. 故 $\mathbf{r}(s) - \mathbf{p}_0 \perp \mathbf{t}$. 因此,

$$\mathbf{r}(s) - \mathbf{p}_0 = \lambda(s) \mathbf{n}(s) + \mu(s) \mathbf{b}(s),$$

其中 $\lambda(s) = \langle \mathbf{r}(s) - \mathbf{p}_0, \mathbf{n} \rangle$, $\mu(s) = \langle \mathbf{r}(s) - \mathbf{p}_0, \mathbf{b} \rangle$ 是光滑函数. 对上式两边求导数, 有

$$\mathbf{t} = -\lambda \kappa \mathbf{t} + (\lambda' - \mu \tau) \mathbf{n} + (\lambda \tau + \mu') \mathbf{b}.$$

故

$$\lambda \kappa = -1, \quad \lambda' - \mu \tau = 0, \quad \lambda \tau + \mu' = 0.$$

因此,

$$\lambda = -\frac{1}{\kappa}, \quad \mu = \frac{1}{\tau} \lambda' = -\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right).$$

所以,

$$\mathbf{r}(s) - \mathbf{p}_0 = -\frac{1}{\kappa} \mathbf{n}(s) - \frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right) \mathbf{b}(s).$$

故

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right)^2 = a^2.$$

这说明: 如果一个弧长参数曲线 $\mathbf{r}(s)$ 的曲率不为常数, 它落在某个球面上的一个充要条件是

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right)\right)^2 = \text{常数}.$$

(2) 另外的证明方法:

假设 κ 不为常数. 定义曲线

$$\tilde{\mathbf{r}}(s) := -\frac{1}{\kappa} \mathbf{n} - \frac{1}{\tau} \frac{d}{ds} \left(\frac{1}{\kappa}\right) \mathbf{b}.$$

根据假设, 这是一个球面曲线. 直接计算并应用假设[参考注 (1)], 有

$$\tilde{\mathbf{r}}'(s) = \mathbf{t}(s).$$

故 s 是 $\tilde{\mathbf{r}}(s)$ 的弧长参数且 $\tilde{\mathbf{r}}(s)$ 的 Frenet 标架为 $\{\tilde{\mathbf{r}}(s); \mathbf{t}, \mathbf{n}, \mathbf{b}\}$. 因此,

$$\tilde{\kappa}(s) = \kappa(s), \quad \tilde{\tau}(s) = \tau(s).$$

根据曲线论基本定理 4.2 [唯一性], 曲线 $\tilde{\mathbf{r}}(s)$ 与 $\mathbf{r}(s)$ 只相差一个刚体运动. 故曲线 $\tilde{\mathbf{r}}(s)$ 也是球面曲线.

16. 设 P_0 是 E^3 中曲线 C 上的点, P 是 C 上 P_0 的邻近点, l 是 P_0 处的切线. 证明:

$$\lim_{P \rightarrow P_0} \frac{2d(P, l)}{d^2(P_0, P)} = \kappa(P_0),$$

这里 d 表示 E^3 的距离.

证明: 设 $\mathbf{r}(s)$ 是曲线 C 的弧长参数表达式且 $\mathbf{r}(s_0) = \overrightarrow{OP_0}$.

方法一: (应用 L'Hôpital 法则)

由 L'Hôpital 法则及 Frenet 公式, 有

$$\begin{aligned} \lim_{s \rightarrow s_0} \frac{\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{n}(s_0) \rangle}{|\mathbf{r}(s) - \mathbf{r}(s_0)|^2} &= \lim_{s \rightarrow s_0} \frac{\langle \mathbf{t}(s), \mathbf{n}(s_0) \rangle}{2\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{t}(s) \rangle} \\ &= \lim_{s \rightarrow s_0} \frac{\kappa(s) \langle \mathbf{n}(s), \mathbf{n}(s_0) \rangle}{2(1 + \langle \mathbf{r}(s) - \mathbf{r}(s_0), \dot{\mathbf{t}}(s) \rangle)} = \frac{1}{2} \kappa(s_0). \end{aligned}$$

所以,

$$\lim_{P \rightarrow P_0} \frac{2d(P, l)}{d^2(P_0, P)} = \lim_{s \rightarrow s_0} \frac{2|\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{n}(s_0) \rangle|}{|\mathbf{r}(s) - \mathbf{r}(s_0)|^2} = \kappa(s_0) = \kappa(P_0).$$

方法二: (应用 Taylor 展式)

由 Frenet 公式, 曲线 $\mathbf{r}(s)$ 在 s_0 处的 Taylor 展式为

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{r}(s_0) + \mathbf{t}(s_0)((s - s_0) + o(s - s_0)^2) + \mathbf{n}(s_0)\left(\frac{1}{2}\kappa(s_0)(s - s_0)^2 + o(s - s_0)^2\right) \\ &\quad + \mathbf{b}(s_0)o(s - s_0)^2. \end{aligned}$$

因此,

$$d(P, l) = |\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{n}(s_0) \rangle| = \frac{1}{2}\kappa(s_0)(s - s_0)^2 + o(s - s_0)^2,$$

$$d^2(P_0, P) = |\mathbf{r}(s) - \mathbf{r}(s_0)|^2 = (s - s_0)^2 + o(s - s_0)^2.$$

故

$$\begin{aligned} \lim_{P \rightarrow P_0} \frac{2d(P, l)}{d^2(P_0, P)} &= \lim_{s \rightarrow s_0} \frac{2|\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{n}(s_0) \rangle|}{|\mathbf{r}(s) - \mathbf{r}(s_0)|^2} \\ &= \lim_{s \rightarrow s_0} \frac{2(\frac{1}{2}\kappa(s_0)(s - s_0)^2 + o(s - s_0)^2)}{(s - s_0)^2 + o(s - s_0)^2} = \kappa(s_0) = \kappa(P_0). \end{aligned}$$

17. 求曲率和挠率满足 $\tau = c\kappa$ (c 为常数, $\kappa > 0$) 的曲线.

解: 设要求解的空间曲线的弧长参数式为 $\mathbf{r} = \mathbf{r}(s)$.

(1) $c = 0$ 时, $\tau = 0$. 故此时曲线落在某个平面上, 可以应用习题 13 的解中叙述过的一般方法求解. 不过, 需要注意的是, 根据定义, 平面曲线的曲率可以取负值, 与空间曲线的曲率定义略有差别. 需要求解的空间曲线作为平面曲线的曲率为 $\pm\kappa$. 因此, 最后求得两类空间曲线:

$$\mathbf{r}(s) = \left(\int_0^s \cos\left(\int_0^u \kappa(t)dt\right)du + a \right) + c_1, \int_0^s \sin\left(\int_0^u \kappa(t)dt\right)du + a + c_2),$$

$$\tilde{\mathbf{r}}(s) = \left(\int_0^s \cos\left(\int_0^u \kappa(t)dt\right)du + a \right) \mathbf{e}_1 - \int_0^s \sin\left(\int_0^u \kappa(t)dt\right)du + a \mathbf{e}_2 + c_2 \mathbf{e}_3.$$

(2) 假设 $c \neq 0$. 由 Frenet 公式, 有

$$\begin{cases} \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \\ \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} + \tau\mathbf{b} \\ \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} \end{cases} \quad (7)$$

作(容许的)参数变换 $\theta(s) = \int_0^s \kappa(t)dt$. 则 $d\theta = \kappa(s)ds$. 方程组 (7) 可以改写为

$$\begin{cases} \frac{d\mathbf{t}(\theta)}{d\theta} = \mathbf{n}(\theta) \\ \frac{d\mathbf{n}(\theta)}{d\theta} = -\mathbf{t}(\theta) + c\mathbf{b}(\theta) \\ \frac{d\mathbf{b}(\theta)}{d\theta} = -c\mathbf{n}(\theta) \end{cases} \quad (8)$$

由此得到

$$\frac{d^2\mathbf{n}(\theta)}{d\theta^2} = -a^2\mathbf{n}(\theta),$$

其中 $a = \sqrt{1+c^2}$. 由常微分方程理论, 此方程有通解

$$\mathbf{n}(\theta) = \cos a\theta \mathbf{e}_1 + \sin a\theta \mathbf{e}_2,$$

其中 $\mathbf{e}_1, \mathbf{e}_2$ 是常向量. 从而, 可以解出方程组 (8) 中的第一式, 得

$$\mathbf{t}(\theta) = \frac{1}{a}(\sin a\theta \mathbf{e}_1 - \cos a\theta \mathbf{e}_2 + c\mathbf{e}_3),$$

其中 $\mathbf{e}_3$ 是常向量. 由方程组 (8) 中的第二式, 有

$$\mathbf{b}(\theta) = -\frac{c}{a}(\sin a\theta \mathbf{e}_1 - \cos a\theta \mathbf{e}_2) + \frac{1}{a}\mathbf{e}_3.$$

由于 $s=0$ 时(即: $\theta(0)=0$ 时), Frenet 标架 $\{\mathbf{r}(0); \mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)\}$ 是右手系的且单位正交. 而易知

$$\begin{pmatrix} \mathbf{t}(0) \\ \mathbf{n}(0) \\ \mathbf{b}(0) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{a} & \frac{c}{a} \\ 1 & 0 & 0 \\ 0 & \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

其中的三阶矩阵是行列式为 1 的正交矩阵. 因此, 须选取 $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 为右手系且单位正交. 最后, 由 $\frac{d\mathbf{r}(s)}{ds} = \mathbf{t}(s)$ 得到所要求的弧长参数曲线:

$$\begin{aligned} \mathbf{r}(\theta) &= \frac{1}{a} \left(\int_0^s \sin(a\theta(t))dt \mathbf{e}_1 - \int_0^s \cos(a\theta(t))dt \mathbf{e}_2 + c \mathbf{e}_3 \right) + \mathbf{v}, \\ &= \frac{1}{\sqrt{1+c^2}} \left(\int_0^s \sin(\sqrt{1+c^2} \int_0^t \kappa(u)du)dt \mathbf{e}_1 - \int_0^s \cos(\sqrt{1+c^2} \int_0^t \kappa(u)du)dt \mathbf{e}_2 + c \mathbf{e}_3 \right) + \mathbf{v} \end{aligned}$$

其中 $\mathbf{v}$ 是常向量.

注: 关键想法: 引入切向量的“转角”参数简化要求解的常微分方程组. 这一点跟平面曲线的情形类似.

18. (1) 设 $\mathbf{r}(t)$ 是平面曲线, 曲率为 $\kappa(t)$, 求曲线 $\tilde{\mathbf{r}}(t) = \mathbf{r}(-t)$ 的曲率;
 (2) 当 $\mathbf{r}(t)$ 是 E^3 的曲线时, 求曲线 $\tilde{\mathbf{r}}(t) = \mathbf{r}(-t)$ 的曲率和挠率.

解: (1) 设 $\mathbf{r}(t) = (x(t), y(t))$, 则 $\tilde{\mathbf{r}}(t) = (\tilde{x}(t), \tilde{y}(t)) = (-x(-t), -y(-t))$. 从而,

$$\tilde{\mathbf{r}}'(t) = (\tilde{x}'(t), \tilde{y}'(t)) = (-x'(-t), -y'(-t)),$$

$$\tilde{\mathbf{r}}''(t) = (\tilde{x}''(t), \tilde{y}''(t)) = (x''(-t), y''(-t)).$$

故由习题 2, 有

$$\tilde{\kappa}(t) = \frac{\tilde{x}'(t)\tilde{y}''(t) - \tilde{x}''(t)\tilde{y}'(t)}{(\tilde{x}'(t)^2 + \tilde{y}'(t)^2)^{\frac{3}{2}}} = -\frac{x'(-t)y''(-t) - x''(-t)y'(-t)}{(x'(-t)^2 + y'(-t)^2)^{\frac{3}{2}}} = -\kappa(-t).$$

- (2) 由于 $\tilde{\mathbf{r}}'(t) = -\mathbf{r}'(-t)$, $\tilde{\mathbf{r}}''(t) = \mathbf{r}''(-t)$, $\tilde{\mathbf{r}}'''(t) = -\mathbf{r}'''(-t)$, 根据习题 5, 有

$$\tilde{\kappa}(t) = \frac{|\tilde{\mathbf{r}}'(t) \wedge \tilde{\mathbf{r}}''(t)|}{|\tilde{\mathbf{r}}'(t)|^3} = \frac{|-\mathbf{r}'(-t) \wedge \mathbf{r}''(-t)|}{|-\mathbf{r}'(-t)|^3} = \frac{|\mathbf{r}'(-t) \wedge \mathbf{r}''(-t)|}{|\mathbf{r}'(-t)|^3} = \kappa(-t),$$

$$\begin{aligned} \tilde{\tau}(t) &= \frac{(\tilde{\mathbf{r}}', \tilde{\mathbf{r}}'', \tilde{\mathbf{r}}''')}{|\tilde{\mathbf{r}}'(t) \wedge \tilde{\mathbf{r}}''(t)|^2} = \frac{(-\mathbf{r}'(-t), \mathbf{r}''(-t), -\mathbf{r}'''(-t))}{|-\mathbf{r}'(-t) \wedge \mathbf{r}''(-t)|^2} \\ &= \frac{(\mathbf{r}'(-t), \mathbf{r}''(-t), \mathbf{r}'''(-t))}{|-\mathbf{r}'(-t) \wedge \mathbf{r}''(-t)|^2} = \tau(-t). \end{aligned}$$

19. 求沿弧长参数曲线 $\mathbf{r}(s)$ 的向量场 $\mathbf{v}(s)$, 同时满足以下各式:

$$\dot{\mathbf{t}}(s) = \mathbf{v}(s) \wedge \mathbf{t}(s),$$

$$\dot{\mathbf{n}}(s) = \mathbf{v}(s) \wedge \mathbf{n}(s),$$

$$\dot{\mathbf{b}}(s) = \mathbf{v}(s) \wedge \mathbf{b}(s).$$

解: (提示: 应用性质 1.1 (1), p.4.)

首先, 向量场 $\mathbf{v} = \mathbf{v}(s)$ 可以表示为

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{t} \rangle \mathbf{t} + \langle \mathbf{v}, \mathbf{n} \rangle \mathbf{n} + \langle \mathbf{v}, \mathbf{b} \rangle \mathbf{b}.$$

一方面, 由性质 1.1 (1), 有

$$\mathbf{t} \wedge (\mathbf{v} \wedge \mathbf{n}) = \langle \mathbf{t}, \mathbf{n} \rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{t} \rangle \mathbf{n} = -\langle \mathbf{v}, \mathbf{t} \rangle \mathbf{n}.$$

另一方面, 由假设及 Frenet 公式, 得

$$\mathbf{t} \wedge (\mathbf{v} \wedge \mathbf{n}) = \mathbf{t} \wedge \dot{\mathbf{n}} = \mathbf{t} \wedge (-\kappa \mathbf{t} + \tau \mathbf{b}) = -\tau \mathbf{n}.$$

故有, $\langle \mathbf{v}, \mathbf{t} \rangle = \tau$.

类似地,

$$0 = \mathbf{n} \wedge (\kappa \mathbf{n}) = \mathbf{n} \wedge \dot{\mathbf{t}} = \mathbf{n} \wedge (\mathbf{v} \wedge \mathbf{t}) = \langle \mathbf{n}, \mathbf{t} \rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{n} \rangle \mathbf{t} = -\langle \mathbf{v}, \mathbf{n} \rangle \mathbf{t}$$

$$-\kappa \mathbf{t} = \mathbf{b} \wedge (\kappa \mathbf{n}) = \mathbf{b} \wedge \dot{\mathbf{t}} = \mathbf{b} \wedge (\mathbf{v} \wedge \mathbf{t}) = \langle \mathbf{b}, \mathbf{t} \rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{b} \rangle \mathbf{t} = -\langle \mathbf{v}, \mathbf{b} \rangle \mathbf{t}$$

从而, 有 $\langle \mathbf{v}, \mathbf{n} \rangle = 0$, $\langle \mathbf{v}, \mathbf{b} \rangle = \kappa$.

因此,

$$\mathbf{v} = \tau \mathbf{t} + \kappa \mathbf{b}.$$

20. 证明: 曲线 $\mathbf{r}(t) = (t + \sqrt{3} \sin t, 2 \cos t, \sqrt{3}t - \sin t)$ 与曲线 $\tilde{\mathbf{r}}(t) = (2 \cos \frac{t}{2}, 2 \sin \frac{t}{2}, -t)$ 是合同的.

证明: 由例 3.2 (p.22), 曲线 $\tilde{\mathbf{r}}(t)$ 是圆柱螺旋线. 从而, $\tilde{\kappa}(t) = \frac{1}{4}$, $\tilde{\tau}(t) = -\frac{1}{4}$.

现在考虑曲线 $\mathbf{r}(t)$. 直接计算, 有

$$\mathbf{r}'(t) = (1 + \sqrt{3} \cos t, -2 \sin t, \sqrt{3} - \cos t), |\mathbf{r}'(t)| = 2\sqrt{2},$$

$$\mathbf{r}''(t) = (-\sqrt{3} \sin t, -2 \cos t, \sin t),$$

$$\mathbf{r}'''(t) = (-\sqrt{3} \cos t, 2 \sin t, \cos t),$$

$$\mathbf{r}'(t) \wedge \mathbf{r}''(t) = (2\sqrt{3} \cos t - 2, -4 \sin t, -2\sqrt{3} - 2 \cos t), |\mathbf{r}'(t) \wedge \mathbf{r}''(t)| = 4\sqrt{2},$$

$$(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = -8.$$

由习题 5 (p.28), $\kappa(t) = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1}{4}$, $\tau(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|^2} = -\frac{1}{4}$. 由定理 4.2 (曲线论基本定理[唯一性], p.25), 曲线 $\mathbf{r}(t)$ 和 $\tilde{\mathbf{r}}(t)$ 相差 E^3 的一个刚体运动, 从而它们是合同的.

注: 一般地, 要证两条正曲率的曲线 $\mathbf{r}(t)$, $\tilde{\mathbf{r}}(t)$ 是合同的, 可以按如下步骤进行:

- (1) 转化为弧长参数曲线 $\mathbf{r}(s)$, $\tilde{\mathbf{r}}(s)$ (定义在同一个参数区间上);
- (2) 比较曲率和挠率: $\kappa(s) = \tilde{\kappa}(s)$, $\tau(s) = \pm \tilde{\tau}(s)$;
- (3) 由定理 4.2 (加上反刚体运动的情形), 得出结论.

21. 证明定理 4.4 (p.27).

定理 4.4 设 $\kappa(s)$ 是连续可微函数, 则

- (1) 存在平面曲线 $\mathbf{r}(t)$, 它以 s 为弧长参数, 以 $\kappa(s)$ 为曲率;
- (2) 上述曲线在相差平面的一个刚体运动的意义下是惟一的.

证明: 经适当修改, 与定理 4.3 的证明基本相同.

习题三

1. 求下列曲面的参数表达式:

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (椭球面);

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (单叶双曲面);

(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ (双叶双曲面);

(4) $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (椭圆抛物面);

(5) $z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$ (双曲抛物面或者马鞍面);

解: 一般地, 可以按照如下步骤得到一个二次曲面的标准方程的参数表达式:

第一步: 平截化归. 将某个变量看作一个常数, 则得到一个二次曲线的方程(椭圆、双曲线或抛物线). 化归为标准方程并得到其参数表达式.

第二步: 消根号整理. 第一步得到的参数表达式一般有二次根号, 可以取适当参数消去根号并整理得到二次曲面的参数表达式.

[上述方法可以推广到某些其它类型曲面. 另外, 一些曲面的参数表达式可以根据其特定的几何性质得到.]

(1) 椭球坐标表示(类似球坐标表示)

$$\mathbf{r}(u, v) = (a \cos u \cos v, b \cos u \sin v, c \sin u) \left(-\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi\right).$$

椭球极投影坐标表示(类似球极投影坐标表示)

$$\mathbf{r}(u, v) = \left(\frac{2u}{\frac{u^2}{a^2} + \frac{v^2}{b^2} + 1}, \frac{2v}{\frac{u^2}{a^2} + \frac{v^2}{b^2} + 1}, \frac{c(\frac{u^2}{a^2} + \frac{v^2}{b^2} - 1)}{\frac{u^2}{a^2} + \frac{v^2}{b^2} + 1}\right) ((u, v) \in \mathbb{R}^2).$$

(2) 多种参数表示:

$$\mathbf{r}(u, v) = (a \sec u \cos v, b \sec u \sin v, c \tan u) \left(-\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi\right);$$

$$\mathbf{r}(u, v) = (a \cosh u \cos v, b \cosh u \sin v, c \sinh u) (u \in \mathbb{R}, 0 < v < 2\pi);$$

$$\mathbf{r}(u, v) = (a(\cos u - v \sin u), b(\sin u + v \cos u), cv) (0 < u < 2\pi, v \in \mathbb{R});$$

$$\mathbf{r}(u, v) = (a\sqrt{1+u^2} \cos v, b\sqrt{1+u^2} \sin v, cu) (u \in \mathbb{R}, 0 < v < 2\pi).$$

(3) 多种参数表示

$$\mathbf{r}(u, v) = (a \tan u \cos v, b \tan u \sin v, c \sec u) \left(-\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi\right)$$

$$\mathbf{r}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u) (u \in \mathbb{R}, 0 < v < 2\pi)$$

$$\mathbf{r}(u, v) = (a\sqrt{u^2-1} \cos v, b\sqrt{u^2-1} \sin v, cu) (|u| > 1, 0 < v < 2\pi).$$

(4)

$$\mathbf{r}(u, v) = \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2}\right) ((u, v) \in \mathbb{R}^2);$$

$$\mathbf{r}(u, v) = (au \cos v, bu \sin v, u^2) (u \in \mathbb{R}, 0 < v < 2\pi).$$

(5)

$$\mathbf{r}(u, v) = \left(u, v, -\frac{u^2}{a^2} + \frac{v^2}{b^2}\right) ((u, v) \in \mathbb{R}^2);$$

$$\mathbf{r}(u, v) = (au \tan v, bu \sec v, u^2) (u \neq 0, 0 < v < 2\pi) \text{ (上半部分)}.$$

$$\mathbf{r}(u, v) = (au \sec v, bu \tan v, -u^2) (u \neq 0, 0 < v < 2\pi) \text{ (下半部分)}.$$

2. (1) $\mathbf{r}(u, v) = (a(u+v), b(u-v), 4uv)$ 是什么曲面?

(2) $\mathbf{r}(u, v) = (au \cosh v, bu \sinh v, u^2)$ 是什么曲面?

解: (1) 此曲面的方程是 $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$, 它是双曲抛物面.

(2) 此曲面的方程也是 $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$, 但 $z \geq 0$, 因此它是双曲抛物面在 xy -平面上的部分.

3. 求 xy -平面的曲线 $\mathbf{r}(t) = (x(t), y(t))$ 沿 E^3 的常方向 $\mathbf{a}$ 平行移动所得的曲面的参数表达式.

解: 此曲面上每点的位置向量是曲线上某点的位置向量和常方向 $\mathbf{a}$ 的某个倍数的和. 故此曲面有参数表达式:

$$\tilde{\mathbf{r}}(u, v) = (x(u), y(u), 0) + v\mathbf{a}.$$

注: 一般地, 若 $\mathbf{r}(t)$ 是一条空间曲线, 则它沿 E^3 的常方向 $\mathbf{a}$ 平行移动所得的曲面的参数表达式是

$$\tilde{\mathbf{r}}(u, v) = \mathbf{r}(u) + v\mathbf{a}.$$

4. 证明: 曲面 $F(\frac{y}{x}, \frac{z}{x}) = 0$ 的任意切平面过原点.

证明: 记 $G(x, y, z) = F(\frac{y}{x}, \frac{z}{x})$. 设点 (x, y, z) 在曲面 $G(x, y, z) = 0$ 上. 则曲面在此点的一个法向量为 (G_x, G_y, G_z) , 其中 $G_x = -\frac{yF_1(\frac{y}{x}, \frac{z}{x}) + zF_2(\frac{y}{x}, \frac{z}{x})}{x^2}$, $G_y = \frac{F_1(\frac{y}{x}, \frac{z}{x})}{x}$, $G_z = \frac{F_2(\frac{y}{x}, \frac{z}{x})}{x}$, 这里 F_i 表示 F 对第 i 个位置的偏导数. 因此曲面在此点的切平面为

$$-(yF_1 + zF_2)(X - x) + xF_1(Y - y) + xF_2(Z - z) = 0.$$

由于

$$-(yF_1 + zF_2)(-x) + xF_1(-y) + xF_2(-z) = 0,$$

故此平面过原点.

5. 设曲面 S 与平面 Π 相交于 P 点, 且 S 位于 Π 的同一侧, 证明: Π 是曲面 S 在 P 点的切平面.

证明: 设曲面 S 的参数表达式为 $\mathbf{r} = \mathbf{r}(u, v)$, $P = \mathbf{r}(u_0, v_0)$. 设 $\mathbf{a}$ 是 Π 的非零法向量且指向曲面所在的一侧. 考虑曲面 S 的高度函数

$$h(u, v) := \langle \mathbf{r}(u, v) - \mathbf{r}(u_0, v_0), \mathbf{a} \rangle.$$

由假设, P 是 $h(u, v)$ 的极小值点, 从而是它的临界点. 故

$$h_u(u_0, v_0) = \langle \mathbf{r}_u(u_0, v_0), \mathbf{a} \rangle = 0,$$

$$h_v(u_0, v_0) = \langle \mathbf{r}_v(u_0, v_0), \mathbf{a} \rangle = 0.$$

即: $\mathbf{a} \perp \mathbf{r}_u(u_0, v_0)$ 且 $\mathbf{a} \perp \mathbf{r}_v(u_0, v_0)$. 从而, $\mathbf{a} \perp \mathbf{n}(u_0, v_0)$. 而平面 Π 过点 P , 因此是 P 点的切平面.

6. 证明: 曲面 S 在 P 点的切空间 $T_P S$ 等于曲面上过 P 点的曲线在 P 点的切向量全体.

证明: 由曲面的切向量的定义, 曲面 S 上过点 P 的曲线在 P 点的切向量在切空间 $T_P S$ 中.

反过来, 设 $P = \mathbf{r}(u_0, v_0)$, $\mathbf{v} = a\mathbf{r}_u(P) + b\mathbf{r}_v(P) \in T_P S$. 则 S 上曲线

$$\mathbf{r}(t) = \mathbf{r}(a(t - t_0) + u_0, b(t - t_0) + v_0)$$

过点 P , 因为 $\mathbf{r}(t_0) = P$. 而曲线 $\mathbf{r}(t)$ 在点 P 的切向量为 $\mathbf{v} = a\mathbf{r}_u(P) + b\mathbf{r}_v(P)$, 故 $\mathbf{v}$ 是曲面 S 在 P 点的一个切向量.

7. 求椭球面的第一基本形式.

解: 由习题 1 (1), 椭球面的一个参数表达式为

$$\mathbf{r}(u, v) = (a \cos u \cos v, b \cos u \sin v, c \sin u) \left(-\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi \right)$$

由

$$\mathbf{r}_u = (-a \sin u \cos v, -b \sin u \sin v, c \cos u), \quad \mathbf{r}_v = (-a \cos u \sin v, b \cos u \cos v, 0),$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = a^2 \sin^2 u \cos^2 v + b^2 \sin^2 u \sin^2 v + c^2 \cos^2 u,$$

$$F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = \frac{a^2 - b^2}{4} \sin 2u \sin 2v,$$

$$G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = a^2 \cos^2 u \sin^2 v + b^2 \cos^2 u \cos^2 v.$$

故第一基本形式

$$\begin{aligned} I(u, v) = & (a^2 \sin^2 u \cos^2 v + b^2 \sin^2 u \sin^2 v + c^2 \cos^2 u) du^2 \\ & + \frac{a^2 - b^2}{2} \sin 2u \sin 2v du dv \\ & + (a^2 \cos^2 u \sin^2 v + b^2 \cos^2 u \cos^2 v) dv^2. \end{aligned}$$

8/14. 求下列曲面的第一、二基本形式:

(1) 柱面: $\mathbf{r}(u, v) = (f(u), g(u), v)$;

(2) 正螺旋面: $\mathbf{r}(u, v) = (u \cos v, u \sin v, bv)$;

(3) 椭圆抛物面: $\mathbf{r}(u, v) = (a(u + v), b(u - v), u^2 + v^2)$.

解: (1) 由

$$\mathbf{r}_u = (f'(u), g'(u), 0), \quad \mathbf{r}_v = (0, 0, 1),$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = f'(u)^2 + g'(u)^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = 1.$$

故第一基本形式

$$I(u, v) = (f'(u)^2 + g'(u)^2) du^2 + dv^2.$$

由

$$\mathbf{r}_u \wedge \mathbf{r}_v = (g'(u), -f'(u), 0),$$

有

$$\mathbf{n} = \frac{1}{\sqrt{f'(u)^2 + g'(u)^2}} (g'(u), -f'(u), 0).$$

而

$$\mathbf{r}_{uu} = (f''(u), g''(u), 0), \quad \mathbf{r}_{uv} = \mathbf{0}, \quad \mathbf{r}_{vv} = \mathbf{0},$$

故

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = \frac{f''(u)g'(u) - f'(u)g''(u)}{\sqrt{f'(u)^2 + g'(u)^2}}, \quad M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = 0, \quad N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = 0.$$

从而, 第二基本形式

$$\text{II}(u, v) = \frac{f''(u)g'(u) - f'(u)g''(u)}{\sqrt{f'(u)^2 + g'(u)^2}} du^2.$$

(2) 由

$$\mathbf{r}_u = (\cos v, \sin v, 0), \quad \mathbf{r}_v = (-u \sin v, u \cos v, b),$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = 1, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = u^2 + b^2.$$

故第一基本形式

$$\text{I}(u, v) = du^2 + (u^2 + b^2)dv^2.$$

由

$$\mathbf{r}_u \wedge \mathbf{r}_v = (b \sin v, -b \cos v, u),$$

有

$$\mathbf{n} = \frac{1}{\sqrt{u^2 + b^2}} (b \sin v, -b \cos v, u).$$

而

$$\mathbf{r}_{uu} = \mathbf{0}, \quad \mathbf{r}_{uv} = (-\sin v, \cos v, 0), \quad \mathbf{r}_{vv} = (-u \cos v, -u \sin v, 0),$$

故

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = 0, \quad M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = -\frac{b}{\sqrt{u^2 + b^2}}, \quad N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = 0.$$

从而, 第二基本形式

$$\text{II}(u, v) = -2 \frac{b}{\sqrt{u^2 + b^2}} du dv.$$

(3) 由

$$\mathbf{r}_u = (a, b, 2u), \quad \mathbf{r}_v = (a, -b, 2v),$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = a^2 + b^2 + 4u^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = a^2 - b^2 + 4uv, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = a^2 + b^2 + 4v^2.$$

故第一基本形式

$$\text{I}(u, v) = (a^2 + b^2 + 4u^2)du^2 + 2(a^2 - b^2 + 4uv)dudv + (a^2 + b^2 + 4v^2)dv^2.$$

由

$$\mathbf{r}_u \wedge \mathbf{r}_v = (2b(u + v), 2a(u - v), -2ab),$$

有

$$\mathbf{n} = \frac{1}{\sqrt{a^2(u - v)^2 + b^2(u + v)^2 + a^2b^2}} (b(u + v), a(u - v), -ab).$$

而

$$\mathbf{r}_{uu} = (0, 0, 2), \quad \mathbf{r}_{uv} = \mathbf{0}, \quad \mathbf{r}_{vv} = (0, 0, 2),$$

故

$$L = N = -\frac{2ab}{\sqrt{a^2(u - v)^2 + b^2(u + v)^2 + a^2b^2}}, \quad M = 0.$$

从而, 第二基本形式

$$\text{II}(u, v) = -\frac{2ab}{\sqrt{a^2(u - v)^2 + b^2(u + v)^2 + a^2b^2}} (du^2 + dv^2).$$

9/15. 求曲面 $z = f(x, y)$ 的第一、二基本形式.

解: 显然, 曲面有参数表示式 $\mathbf{r}(x, y) = (x, y, f(x, y))$.

由

$$\mathbf{r}_x = (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y),$$

有

$$E = \langle \mathbf{r}_x, \mathbf{r}_x \rangle = 1 + f_x^2, \quad F = \langle \mathbf{r}_x, \mathbf{r}_y \rangle = f_x f_y, \quad G = \langle \mathbf{r}_y, \mathbf{r}_y \rangle = 1 + f_y^2.$$

故第一基本形式

$$I(x, y) = (1 + f_x^2)dx^2 + 2f_x f_y dx dy + (1 + f_y^2)dy^2.$$

由

$$\mathbf{r}_x \wedge \mathbf{r}_y = (-f_x, -f_y, 1),$$

有

$$\mathbf{n} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}(-f_x, -f_y, 1).$$

而

$$\mathbf{r}_{xx} = (0, 0, f_{xx}), \quad \mathbf{r}_{xy} = (0, 0, f_{xy}), \quad \mathbf{r}_{yy} = (0, 0, f_{yy}),$$

故

$$\begin{aligned} L &= \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \\ M &= \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \\ N &= \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}. \end{aligned}$$

从而, 第二基本形式

$$II(x, y) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}(f_{xx}dx^2 + 2f_{xy}dx dy + f_{yy}dy^2).$$

10. 设 $F_\lambda(x, y, z) = \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda}$ ($a > b > c > 0$). 当 $\lambda \in (-\infty, c)$ 时, $F_\lambda = 1$ 给出了一族椭球面; $\lambda \in (c, b)$ 时, $F_\lambda = 1$ 给出了一族单叶双曲面; $\lambda \in (b, a)$ 时, $F_\lambda = 1$ 给出了一族双叶双曲面. 证明: 对 E^3 中任意一点 $P = (x, y, z)$ ($xyz \neq 0$), 恰有分别属于这三族曲面的三个二次曲面过 P 点, 且它们在 P 点相互正交.

证明: 设点 $P = (x, y, z)$ ($xyz \neq 0$) 在曲面 $S_\lambda: \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$ 上. 考虑关于 λ 的三次多项式

$$f(\lambda) = (b-\lambda)(c-\lambda)x^2 + (a-\lambda)(c-\lambda)y^2 + (a-\lambda)(b-\lambda)z^2 - (a-\lambda)(b-\lambda)(c-\lambda).$$

注意到 $f(-\infty) < 0$, $f(c) > 0$, $f(b) < 0$ 且 $f(a) > 0$. 故 $f(\lambda)$ 在区间 $(-\infty, c)$, (c, b) , (b, a) 上恰好各有一根, 记为 λ_i ($1 \leq i \leq 3$). 下面只需证明它们对应的曲面 S_{λ_i} 在 P 点处相互正交, 即: 在 P 点处这三个曲面的法向量相互正交.

曲面 S_{λ_i} 有非零法向量 $\mathbf{n}_i = (\frac{x}{a-\lambda_i}, \frac{y}{b-\lambda_i}, \frac{z}{c-\lambda_i})$. 对于 $i \neq j$,

$$\langle \mathbf{n}_i, \mathbf{n}_j \rangle = \frac{x^2}{(a-\lambda_i)(a-\lambda_j)} + \frac{y^2}{(b-\lambda_i)(b-\lambda_j)} + \frac{z^2}{(c-\lambda_i)(c-\lambda_j)}$$

$$= \frac{1}{\lambda_i - \lambda_j} \left[\left(\frac{x^2}{a - \lambda_i} - \frac{x^2}{a - \lambda_j} \right) + \left(\frac{y^2}{b - \lambda_i} - \frac{y^2}{b - \lambda_j} \right) + \left(\frac{z^2}{c - \lambda_i} - \frac{z^2}{c - \lambda_j} \right) \right] = 0.$$

即: $\mathbf{n}_i$ 两两正交.

11. 设 (x, y) 是曲面 $\mathbf{r}(u, v)$ 的另一组参数, 问: $\mathbf{r}_u \wedge \mathbf{r}_v$ 与 $\mathbf{r}_x \wedge \mathbf{r}_y$ 的指向是否相同?

解: 由于

$$\mathbf{r}_u \wedge \mathbf{r}_v = \frac{\partial(u, v)}{\partial(x, y)} \mathbf{r}_x \wedge \mathbf{r}_y,$$

故当 $\frac{\partial(u, v)}{\partial(x, y)} > 0$ 时, $\mathbf{r}_u \wedge \mathbf{r}_v$ 与 $\mathbf{r}_x \wedge \mathbf{r}_y$ 的指向相同; 当 $\frac{\partial(u, v)}{\partial(x, y)} < 0$ 时, $\mathbf{r}_u \wedge \mathbf{r}_v$ 与 $\mathbf{r}_x \wedge \mathbf{r}_y$ 的指向相反.

12. 使 $F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0$ 的参数 (u, v) 称为曲面的正交参数系. 给定一个曲面 S 以及它的一个参数表示 $\mathbf{r} = \mathbf{r}(u, v)$, 证明: 对曲面 S 上任意一点 $P_0 = \mathbf{r}(u, v)$, 存在 P_0 的邻域 D 以及 D 的新参数 (s, t) , 使得 (s, t) 是曲面 S 的正交参数系.

证明: 下面证明更一般的一个结果:

设曲面 S 上有两个处处线性无关的向量场 $\mathbf{a}(u, v)$, $\mathbf{b}(u, v)$. 则对任意点 $P \in S$, 存在 P 的邻域 $U \subseteq S$ 及 U 上的新参数 (s, t) , 使得新参数的坐标切向量分别与 $\mathbf{a}$, $\mathbf{b}$ 平行.

注: 将上述结果应用于曲面 S 上任意两个正交的向量场就证明了此习题. 正交向量场的取法有任意性, 一个自然的取法是将坐标切向量正交化.

由于向量场 $\mathbf{a}, \mathbf{b}$ 是处处线性无关的, 可以设

$$\begin{cases} \mathbf{a} = a_1 \mathbf{r}_u + a_2 \mathbf{r}_v \\ \mathbf{b} = b_1 \mathbf{r}_u + b_2 \mathbf{r}_v \end{cases} \quad (9)$$

记 $d = a_1 b_2 - a_2 b_1 \neq 0$. 假设存在局部的(容许的)参数变换 $u = u(s, t)$, $v = v(s, t)$ 使得 $\mathbf{r}_s // \mathbf{a}$, $\mathbf{r}_t // \mathbf{b}$. 则存在函数 λ, μ 使得 $\mathbf{r}_s = \lambda \mathbf{a} = \lambda a_1 \mathbf{r}_u + \lambda a_2 \mathbf{r}_v$, $\mathbf{r}_t = \mu \mathbf{b} = \mu b_1 \mathbf{r}_u + \mu b_2 \mathbf{r}_v$, 其中 λ, μ 均恒不为 0. 则此变换的 Jacobi 矩阵为 $J = \begin{pmatrix} \lambda a_1 & \lambda a_2 \\ \mu b_1 & \mu b_2 \end{pmatrix}$. 由于 $|J| = \lambda \mu d \neq 0$, J 是处处可逆的. 由反函数存在定理, 此变换有逆, 记为 $s = s(u, v)$, $t = t(u, v)$, 其 Jacobi 矩阵为 $J^{-1} = \frac{1}{\lambda \mu d} \begin{pmatrix} \mu b_2 & -\mu b_1 \\ -\lambda a_2 & \lambda a_1 \end{pmatrix}$. 因此,

$$ds = \frac{1}{\lambda d} (b_2 du - b_1 dv),$$

$$dt = \frac{1}{\mu d} (-a_2 du + a_1 dv).$$

由这两个微分方程知, 满足条件的局部新参数 (s, t) 存在当且仅当一次微分式 $\xi = b_2 du - b_1 dv$, $\eta = -a_2 du + a_1 dv$ 在局部上存在积分因子, 即: 存在(积分因子)函数 f, g 使得 $f\xi, g\eta$ 均为全微分.

只需证明: 对于定义在平面区域 D 上的两个可微函数 $f(x, y), g(x, y)$ 和点 $(x_0, y_0) \in D$, 若 $(f(x_0, y_0), g(x_0, y_0)) \neq (0, 0)$, 则存在 (x_0, y_0) 的一个邻域 $U \subseteq D$ 及 U 上的可微函数 $\rho(x, y)$ 使得 ρ 是一次微分式 $f dx + g dy$ 的积分因子.

不妨设 $g(x_0, y_0) \neq 0$. 则在 (x_0, y_0) 的一个邻域内, $g(x, y) \neq 0$. 在此邻域内, 考虑关于变量 x 的常微分方程 $f(x, y) dx + g(x, y) dy = 0$, 即: $\frac{dy}{dx} = -\frac{f(x, y)}{g(x, y)}$. 由常微分方程关于初值的依赖性, 存在 x_0 的邻域 I 和 y_0 的邻域 J 使得 $I \times J \subseteq D$ 且

对于给定的 $\tilde{y} \in J$, 此常微分方程有惟一可微解 $y = \phi(x, \tilde{y})$, $x \in I$, 即: $\frac{\partial \phi(x, \tilde{y})}{\partial x} = -\frac{f(x, \phi(x, \tilde{y}))}{g(x, \phi(x, \tilde{y}))}$ 且 $\phi(x_0, \tilde{y}) = \tilde{y}$. 由于 $\phi(x, \tilde{y})$ 关于 $\tilde{y}$ 可微, 故 $\frac{\partial \phi(x, \tilde{y})}{\partial \tilde{y}}|_{x=x_0} = \frac{\partial \phi(x_0, \tilde{y})}{\partial \tilde{y}} =$

1. 从而, 存在 x_0 的邻域 $\bar{I} \subseteq I$ 使得对于任意 $(x, \tilde{y}) \in \bar{I} \times J$, 有 $\frac{\partial \phi(x, \tilde{y})}{\partial \tilde{y}} \neq 0$. 故对函数组 $x = \tilde{x}$, $y = \phi(x, \tilde{y})$ 应用反函数定理, 存在区域 $\tilde{I} \times \tilde{J} \subseteq I \times J$ 上的反函数组 $\tilde{x} = x$, $\tilde{y} = \psi(x, y)$.

由

$$dy = -\frac{\partial \phi(x, \tilde{y})}{\partial x} dx + \frac{\partial \phi(x, \tilde{y})}{\partial \tilde{y}} d\tilde{y} = -\frac{f(x, \phi(x, \tilde{y}))}{g(x, \phi(x, \tilde{y}))} dx + \frac{\partial \phi(x, \tilde{y})}{\partial \tilde{y}} d\tilde{y},$$

有

$$d\tilde{y} = \frac{1}{\frac{\partial \phi(x, \tilde{y})}{\partial \tilde{y}} g(x, y)} (f(x, y) dx + g(x, y) dy) = \frac{1}{\frac{\phi(x, \psi(x, y))}{\partial \tilde{y}} g(x, y)} (f(x, y) dx + g(x, y) dy).$$

故 $\rho(x, y) := \frac{1}{\frac{\phi(x, \psi(x, y))}{\partial \tilde{y}} g(x, y)}$ 是 $f(x, y) dx + g(x, y) dy$ 在区域 $\tilde{I} \times \tilde{J}$ 上的积分因子.

13. 在曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 上一点, 由方程 $P\lambda^2 + 2Q\lambda\mu + R\mu^2 = 0$ 确定两个切方向. 证明: 这两个切方向相互正交的充要条件是 $ER - 2FQ + GP = 0$.

证明: 设非零向量 $(\lambda_1, \mu_1), (\lambda_2, \mu_2)$ 是方程 $P\lambda^2 + 2Q\lambda\mu + R\mu^2 = 0$ 的不平行的两个解. 则曲面 S 的切向量 $\mathbf{v}_1 := \lambda_1 \mathbf{r}_u + \mu_1 \mathbf{r}_v$ 与 $\mathbf{v}_2 := \lambda_2 \mathbf{r}_u + \mu_2 \mathbf{r}_v$ 正交等价于

$$0 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \lambda_1 \lambda_2 E + (\lambda_1 \mu_2 + \lambda_2 \mu_1) F + \mu_1 \mu_2 G.$$

情形 1: $\mu_1 \mu_2 = 0$. 不妨设 $\mu_1 = 0$, 则 $\mu_2 \neq 0$, 这是因为 (λ_1, μ_1) 与 (λ_2, μ_2) 不平行. 故 $P\lambda_1^2 = 0$. 由 $\mathbf{v}_1 \neq \mathbf{0}$, 有 $\lambda_1 \neq 0$, 从而, $P = 0$. 因此, $\mathbf{v}_1 \perp \mathbf{v}_2 \Leftrightarrow \lambda_1(\lambda_2 E + \mu_2 F) = 0 \Leftrightarrow \lambda_2 E + \mu_2 F = 0 \Leftrightarrow \frac{\lambda_2}{\mu_2} = -\frac{F}{E}$. 而由 $2Q\lambda_2 + R\mu_2 = 0$, 知 $\frac{\lambda_2}{\mu_2} = -\frac{R}{2Q}$. 故 $\mathbf{v}_1 \perp \mathbf{v}_2 \Leftrightarrow -\frac{F}{E} = -\frac{R}{2Q} \Leftrightarrow ER - 2FQ = 0$.

情形 2: $\mu_1 \mu_2 \neq 0$. 由 Vieta 定理, 有 $\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} = -\frac{2Q}{P}$, $\frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} = \frac{R}{P}$. 因此, $\mathbf{v}_1 \perp \mathbf{v}_2 \Leftrightarrow \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} E + (\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}) F + G = 0 \Leftrightarrow \frac{R}{P} E - \frac{2Q}{P} F + G = 0 \Leftrightarrow ER - 2FQ + GP = 0$.

16 求曲面 $F(x, y, z) = 0$ 的第一、二基本形式.

解: 设点 $P = (x, y, z)$ 在曲面 $S: F(x, y, z) = 0$ 上. 由 $\nabla F(x, y, z) \neq \mathbf{0}$, 不妨设在点 P , $F_z \neq 0$. 则在 P 的一个邻域 U 内, $F_z \neq 0$ 且 S 有显式表达 $z = f(x, y)$. 从而, 在 U 内, S 有参数表达式 $\mathbf{r}(x, y) = (x, y, f(x, y))$. 由于 $f_x = -\frac{F_x}{F_z}$, $f_y = -\frac{F_y}{F_z}$, 应用习题 9/15, 则在 U 内, 曲面 S 的第一基本形式

$$\begin{aligned} I(x, y) &= (1 + f_x^2) dx^2 + 2f_x f_y dx dy + (1 + f_y^2) dy^2 \\ &= (1 + \frac{F_x^2}{F_z^2}) dx^2 + 2 \frac{F_x F_y}{F_z^2} dx dy + (1 + \frac{F_y^2}{F_z^2}) dy^2. \end{aligned}$$

由于

$$\begin{aligned} f_{xx} &= \frac{-F_z^2 F_{xx} + 2F_x F_z F_{xz} - F_x^2 F_{zz}}{F_z^3}, \\ f_{xy} &= \frac{-F_z^2 F_{xy} + F_y F_z F_{xz} + F_x F_z F_{yz} - F_x F_y F_{zz}}{F_z^3}, \\ f_{yy} &= \frac{-F_z^2 F_{yy} + 2F_y F_z F_{yz} - F_y^2 F_{zz}}{F_z^3}, \end{aligned}$$

应用习题 9/15, 曲面 S 在 U 内的第二基本形式

$$\begin{aligned} \text{II}(x, y) &= \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}(f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2) \\ &= \frac{\text{sgn}(F_z)}{F_z^2 \sqrt{F_x^2 + F_y^2 + F_z^2}} [(-F_z^2 F_{xx} + 2F_x F_z F_{xz} - F_x^2 F_{zz})dx^2 \\ &\quad + 2(-F_z^2 F_{xy} + F_y F_z F_{xz} + F_x F_z F_{yz} - F_x F_y F_{zz})dxdy \\ &\quad + (-F_z^2 F_{yy} + 2F_y F_z F_{yz} - F_y^2 F_{zz})dy^2]. \end{aligned}$$

17. 证明: 在曲面的任意一点, 任何两个相互正交的切方向的法曲率之和为常数.

证明: 设曲面 S 在其上任意一点 P 的主曲率为 k_1, k_2 , 而 v_1, v_2 是 P 点处相互垂直的任意两个切向量. 由 Euler 公式, 有

$$\begin{aligned} k_n(v_1) + k_n(v_2) &= k_1 \cos^2 \theta + k_2 \sin^2 \theta + k_1 \cos^2(\theta \pm \frac{\pi}{2}) + k_2 \sin^2(\theta \pm \frac{\pi}{2}) \\ &= k_1 \cos^2 \theta + k_2 \sin^2 \theta + k_1 \cos^2 \theta + k_2 \sin^2 \theta = k_1 + k_2 = 2H(P), \end{aligned}$$

是点 P 处的常数.

18. 设曲面 S 由方程 $x^2 + y^2 - f(z) = 0$ 给定, f 满足 $f(0) = 0, f'(0) \neq 0$. 证明: S 在点 $(0, 0, 0)$ 的法曲率为常数.

证明: 由 $f(0) = 0$ 知, 原点 $O \in S$. 记 $F(x, y, z) = x^2 + y^2 - f(z)$. 则显然有 $F_z(O) = -f'(0) \neq 0$. 故在 O 的某领域内, 隐函数 $F = 0$ 有显式表达 $z = g(x, y)$. 从而, 在 O 的某领域内, 曲面 S 有参数表达式 $\mathbf{r}(x, y) = (x, y, g(x, y))$. 直接计算, 有

$$\begin{aligned} \mathbf{r}_x &= (1, 0, g_x) = (1, 0, \frac{2x}{f'(z)}), \\ \mathbf{r}_y &= (0, 1, g_y) = (0, 1, \frac{2y}{f'(z)}). \end{aligned}$$

故

$$\mathbf{r}_x(O) = (1, 0, 0), \quad \mathbf{r}_y(O) = (0, 1, 0).$$

由此, S 在 O 点的切平面是 xy -平面. 由

$$\mathbf{r}_x \wedge \mathbf{r}_y = (-\frac{2x}{f'(z)}, -\frac{2y}{f'(z)}, 1),$$

有

$$\mathbf{n} = \frac{\text{sgn}(f'(z))}{\sqrt{4x^2 + 4y^2 + f'(z)^2}}(-2x, -2y, f'(z)).$$

而

$$\begin{aligned} \mathbf{r}_{xx} &= (0, 0, \frac{2f'(z)^2 - 4x^2 f''(z)}{f'(z)^3}), \\ \mathbf{r}_{xy} &= (0, 0, -\frac{4xy f''(z)}{f'(z)^3}), \\ \mathbf{r}_{yy} &= (0, 0, \frac{2f'(z)^2 - 4y^2 f''(z)}{f'(z)^3}). \end{aligned}$$

故

$$\begin{aligned} L &= \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \operatorname{sgn}(f'(z)) \frac{2f'(z)^2 - 4x^2 f''(z)}{f'(z)^2 \sqrt{4x^2 + 4y^2 + f'(z)^2}}, \\ M &= \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = -\operatorname{sgn}(f'(z)) \frac{4xy f''(z)}{f'(z)^2 \sqrt{4x^2 + 4y^2 + f'(z)^2}}, \\ N &= \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \operatorname{sgn}(f'(z)) \frac{2f'(z)^2 - 4y^2 f''(z)}{f'(z)^2 \sqrt{4x^2 + 4y^2 + f'(z)^2}}. \end{aligned}$$

从而,

$$L(O) = \frac{2}{f'(0)}, \quad M(O) = 0, \quad N(O) = \frac{2}{f'(0)}.$$

对于 S 在 O 点的任意单位切向量 $v = (\cos \theta, \sin \theta, 0)$, 法曲率

$$k_n(v) = L(O) \cos^2 \theta + 2M(O) \cos \theta \sin \theta + N(O) \sin^2 \theta = \frac{2}{f'(0)},$$

是常数. 从而, 原点处的法曲率为常数 $\frac{2}{f'(0)}$.

注: 曲面 S 是 xz -平面上的曲线 $(\sqrt{f(u)}, f(u))(u \geq 0)$ 绕 z -轴旋转得到的旋转曲面. 直观上看, 在 O 点法曲率有旋转不变性.

19. 定义 $\text{III} = \langle d\mathbf{n}, d\mathbf{n} \rangle$ 为曲面的第三基本形式, 证明: $K\text{I} - 2H\text{II} + \text{III} = 0$.

证明: 方法一:

设曲面 S 的参数表达式为 $\mathbf{r} = \mathbf{r}(u, v)$. (注意到要证的性质是局部的且与同定向的参数系选取无关, 故只需对曲面的一个参数式证明即可.) 由 Weingarten 方程

$$\begin{aligned} \mathbf{n}_u &= \frac{MF - LG}{EG - F^2} \mathbf{r}_u + \frac{LF - ME}{EG - F^2} \mathbf{r}_v, \\ \mathbf{n}_v &= \frac{NF - MG}{EG - F^2} \mathbf{r}_u + \frac{MF - NE}{EG - F^2} \mathbf{r}_v. \end{aligned}$$

有

$$\begin{aligned} \langle \mathbf{n}_u, \mathbf{n}_u \rangle &= \frac{L^2 G - 2LMF + M^2 E}{EG - F^2}, \\ \langle \mathbf{n}_u, \mathbf{n}_v \rangle &= \frac{M(LG + NE) - (LN + M^2)F}{EG - F^2}, \\ \langle \mathbf{n}_v, \mathbf{n}_v \rangle &= \frac{M^2 G - 2MNF + N^2 E}{EG - F^2}. \end{aligned}$$

因此,

$$\begin{aligned} \text{III}(u, v) &= \frac{L^2 G - 2LMF + M^2 E}{EG - F^2} du^2 + 2 \frac{M(LG + NE) - (LN + M^2)F}{EG - F^2} dudv \\ &\quad + \frac{M^2 G - 2MNF + N^2 E}{EG - F^2} dv^2 \end{aligned}$$

设 $K\text{I} - 2H\text{II} + \text{III} = fdu^2 + 2gdudv + hdv^2$, 则

$$f = KE - 2HL + \frac{L^2 G + M^2 E - 2LMF}{EG - F^2} = 0,$$

$$g = KF - 2HM + \frac{M(LG + NE) - (LN + M^2)F}{EG - F^2} = 0,$$

$$h = KG - 2HN + \frac{M^2G + N^2E - 2MNF}{EG - F^2} = 0.$$

因此,

$$KI - 2HII + III = 0.$$

方法二:

设 $\mathcal{W}$ 为曲面 S 的切空间的 Weingarten 变换. 由 $\mathcal{W}(\mathbf{r}_u) = -\mathbf{n}_u$, $\mathcal{W}(\mathbf{r}_v) = -\mathbf{n}_v$, 有

$$\mathcal{W}(d\mathbf{r}) = -d\mathbf{n}.$$

由于 Weingarten 变换是对称的, 故

$$III = \langle \mathcal{W}(d\mathbf{r}), \mathcal{W}(d\mathbf{r}) \rangle = \langle \mathcal{W}^2(d\mathbf{r}), d\mathbf{r} \rangle.$$

而 Weingarten 变换 $\mathcal{W}$ 的特征多项式为 $t^2 - 2Ht + K = 0$, 故由 Cayley-Hamilton 定理, 有

$$\mathcal{W}^2 - 2H\mathcal{W} + K\mathcal{I}_2 = 0,$$

其中 $\mathcal{I}_2$ 是二维恒等变换. 从而,

$$\begin{aligned} KI - 2HII + III &= \langle \mathcal{W}^2(d\mathbf{r}), d\mathbf{r} \rangle - 2H \langle \mathcal{W}(d\mathbf{r}), d\mathbf{r} \rangle + K \langle d\mathbf{r}, d\mathbf{r} \rangle \\ &= \langle (\mathcal{W}^2 - 2H\mathcal{W} + K\mathcal{I})(d\mathbf{r}), d\mathbf{r} \rangle = 0. \end{aligned}$$

注: (1) 方法一中, 若取 (u, v) 为正交曲率参数, 则证明更为简单. 此时, $F = M = 0$. (参考习题 26, 29.) 主曲率

$$k_1 = \frac{L}{E}, \quad k_2 = \frac{N}{G}.$$

注意到平均曲率 $H = \frac{1}{2}(k_1 + k_2)$, Gauss 曲率 $G = k_1k_2$. 由 Weingarten 方程,

$$\mathbf{n}_u = -k_1\mathbf{r}_u, \quad \mathbf{n}_v = -k_2\mathbf{r}_v.$$

故

$$\begin{aligned} I(u, v) &= Edu^2 + Gdv^2, \\ II(u, v) &= k_1Edu^2 + k_2Gdv^2, \\ III(u, v) &= k_1^2Edu^2 + k_2^2Gdv^2. \end{aligned}$$

从而,

$$\begin{aligned} &KI - 2HII + III \\ &= k_1k_2(Edu^2 + Gdv^2) - (k_1 + k_2)(k_1Edu^2 + k_2Gdv^2) + k_1^2Edu^2 + k_2^2Gdv^2 = 0. \end{aligned}$$

(2) 设第一、二、三基本形式对应的矩阵分别是 A, B, C . 由 Weingarten 方程

$$\begin{pmatrix} -\mathbf{n}_u \\ -\mathbf{n}_v \end{pmatrix} = BA^{-1} \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix},$$

有

$$\begin{aligned} C &= \left\langle \begin{pmatrix} -\mathbf{n}_u \\ -\mathbf{n}_v \end{pmatrix}, (-\mathbf{n}_u \ -\mathbf{n}_v) \right\rangle = \left\langle BA^{-1} \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix}, (\mathbf{r}_u \ \mathbf{r}_v) A^{-1}B \right\rangle \\ &= BA^{-1}AA^{-1}B = BA^{-1}B = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \end{aligned}$$

故

$$III(u, v) = (du \ dv) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

20. 设曲面 S_1 和 S_2 的交线 C 的曲率为 κ , 曲线 C 在曲面 S_i 上的法曲率为 k_i ($i = 1, 2$); 若沿 C , S_1 和 S_2 法向的夹角为 θ , 证明:

$$\kappa^2 \sin^2 \theta = k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta.$$

证明: 设曲线 $C: \mathbf{r}(s)$ 以弧长为参数, $\mathbf{n}_i(s)$ 是曲面 S_i 沿曲线 C 的法向量. 由定义, 曲面 S_i 沿曲线 C 的切向量的法曲率 $k_i = k_i(s) = \langle \dot{\mathbf{t}}(s), \mathbf{n}_i(s) \rangle$. 则

$$\begin{aligned} & k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta \\ &= \langle \dot{\mathbf{t}}(s), \mathbf{n}_1(s) \rangle^2 + \langle \dot{\mathbf{t}}(s), \mathbf{n}_2(s) \rangle^2 - 2\langle \dot{\mathbf{t}}(s), \mathbf{n}_1(s) \rangle \langle \dot{\mathbf{t}}(s), \mathbf{n}_2(s) \rangle \cos(\widehat{\mathbf{n}_1, \mathbf{n}_2}) \\ &= |\langle \dot{\mathbf{t}}(s), \mathbf{n}_1(s) \rangle \mathbf{n}_2 - \langle \dot{\mathbf{t}}(s), \mathbf{n}_2(s) \rangle \mathbf{n}_1|^2 \\ &= |\dot{\mathbf{t}}(s) \wedge (\mathbf{n}_2(s) \wedge \mathbf{n}_1(s))|^2 \\ &= |\dot{\mathbf{t}}(s) \wedge (\mathbf{t}(s))|^2 \sin^2 \theta \\ &= \kappa(s)^2 |\mathbf{n}(s) \wedge (\mathbf{t}(s))|^2 \sin^2 \theta \\ &= \kappa(s)^2 \sin^2 \theta. \end{aligned} \tag{10}$$

这里第四个等式用到了 $\mathbf{n}_2(s) \wedge \mathbf{n}_1(s) = \pm \mathbf{t} \sin \theta$, 这由外积的定义得到.

21. 求下列曲面的 Gauss 曲率和平均曲率.

(1) 单叶双曲面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$;

(2) 环面 $\mathbf{r}(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u) (0 < r < R)$.

解: (1) 一个参数表达式为

$$\mathbf{r}(u, v) = (a \sec u \cos v, b \sec u \sin v, c \tan u) \left(-\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi \right).$$

由

$$\mathbf{r}_u = (a \sec u \tan u \cos v, b \sec u \tan u \sin v, c \sec^2 u),$$

$$\mathbf{r}_v = (-a \sec u \sin v, b \sec u \cos v, 0),$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = a^2 \sec^2 u \tan^2 u \cos^2 v + b^2 \sec^2 u \tan^2 u \sin^2 v + c^2 \sec^4 u,$$

$$F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = (b^2 - a^2) \sec^2 u \tan u \sin v \cos v,$$

$$G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = a^2 \sec^2 u \sin^2 v + b^2 \sec^2 u \cos^2 v.$$

又

$$\mathbf{r}_u \wedge \mathbf{r}_v = (-bc \sec^3 u \cos v, -ac \sec^3 u \sin v, ab \sec^2 u \tan u),$$

$$\mathbf{n} = \frac{1}{\sqrt{c^2(a^2 \sin^2 v + b^2 \cos^2 v) + a^2 b^2 \sin^2 u}} (-bc \cos v, -ac \sin v, ab \sin u),$$

$$\mathbf{r}_{uu} = (a \sec u (\sec^2 u + \tan^2 u) \cos v, b \sec u (\sec^2 u + \tan^2 u) \sin v, 2c \sec^2 u \tan u),$$

$$\mathbf{r}_{uv} = (-a \sec u \tan u \sin v, b \sec u \tan u \cos v, 0),$$

$$\mathbf{r}_{vv} = (-a \sec u \cos v, -b \sec u \sin v, 0),$$

有

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = -\frac{abc \sec u}{\sqrt{c^2(a^2 \sin^2 v + b^2 \cos^2 v) + a^2 b^2 \sin^2 u}},$$

$$M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = 0,$$

$$N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = \frac{abc \sec u}{\sqrt{c^2(a^2 \sin^2 v + b^2 \cos^2 v) + a^2 b^2 \sin^2 u}}.$$

故平均曲率

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{abc \cos u (a^2(\sin^2 u \cos^2 v - \cos^2 u \sin^2 v) + b^2(\sin^2 u \sin^2 v - \cos^2 u \cos^2 v) + c^2)}{2(a^2 b^2 \sin^2 u \cos^2 u + a^2 c^2 \sin^2 v + b^2 c^2 \cos^2 v) \sqrt{a^2 b^2 \sin^2 u + c^2(a^2 \sin^2 v + b^2 \cos^2 v)}},$$

Gauss 曲率

$$G = \frac{LN - M^2}{EG - F^2} = \frac{-a^2 b^2 c^2 \cos^4 u}{(a^2 b^2 \sin^2 u \cos^2 u + a^2 c^2 \sin^2 v + b^2 c^2 \cos^2 v) \sqrt{a^2 b^2 \sin^2 u + c^2(a^2 \sin^2 v + b^2 \cos^2 v)}}.$$

(2) 环面的一个参数表达式为

$$\mathbf{r}(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u) (0 < r < R).$$

由

$$\begin{aligned} \mathbf{r}_u &= (-r \sin u \cos v, -r \sin u \sin v, r \cos u), \\ \mathbf{r}_v &= (-(R + r \cos u) \sin v, (R + r \cos u) \cos v, 0), \end{aligned}$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = r^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = (R + r \cos u)^2.$$

又

$$\begin{aligned} \mathbf{r}_u \wedge \mathbf{r}_v &= (-r(R + r \cos u) \cos u \cos v, -r(R + r \cos u) \cos u \sin v, -r(R + r \cos u) \sin u), \\ \mathbf{n} &= (-\cos u \cos v, -\cos u \sin v, -\sin u), \\ \mathbf{r}_{uu} &= (-r \cos u \cos v, -r \cos u \sin v, -r \sin u), \\ \mathbf{r}_{uv} &= (r \sin u \sin v, -r \sin u \cos v, 0), \\ \mathbf{r}_{vv} &= (-(R + r \cos u) \cos v, -(R + r \cos u) \sin v, 0), \end{aligned}$$

有

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = r^2, \quad M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = 0, \quad N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = (R + r \cos u) \cos u.$$

故平均曲率

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{1}{2} \left(1 + \frac{\cos u}{R + r \cos u} \right),$$

Gauss 曲率

$$G = \frac{LN - M^2}{EG - F^2} = \frac{\cos u}{R + r \cos u}.$$

从而, 两个主曲率为

$$k_1 = 1, \quad k_2 = \frac{\cos u}{R + r \cos u}.$$

22. 求曲面 $z = f(x, y)$ 的平均曲率和 Gauss 曲率.

解: 由习题 9/15, 平均曲率

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2)}{2(1 + f_x^2 + f_y^2)^{\frac{3}{2}}},$$

Gauss 曲率

$$K = \frac{LN - M^2}{EG - F^2} = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

23. 求曲面 $\mathbf{r}(u, v) = (u, v, u^2 + v^2)$ 的椭圆点、双曲点和抛物点.

解: 由

$$\mathbf{r}_u = (1, 0, 2u), \quad \mathbf{r}_v = (0, 1, 2v),$$

得

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = 1 + 4u^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 4uv, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = 1 + 4v^2.$$

由 $\mathbf{r}_u \wedge \mathbf{r}_v = (-2u, -2v, 1)$, 有 $\mathbf{n} = \frac{1}{\sqrt{1+4u^2+4v^2}}(-2u, -2v, 1)$. 而又

$$\mathbf{r}_{uu} = (0, 0, 2), \quad \mathbf{r}_{uv} = \mathbf{0}, \quad \mathbf{r}_{vv} = (0, 0, 2),$$

有

$$L = N = \frac{2}{\sqrt{1+4u^2+4v^2}}, \quad M = 0.$$

故

$$LN - M^2 = \frac{4}{1+4u^2+4v^2} > 0.$$

所以, 曲面上所有点都是椭圆点, 没有双曲点和抛物点.

24. 求曲面 $\mathbf{r}(u, v) = (u^3, v^3, u + v)$ 上的抛物点的轨迹.

解: 由

$$\mathbf{r}_u = (3u^2, 0, 1), \quad \mathbf{r}_v = (0, 3v^2, 1),$$

知

$$\mathbf{r}_u \wedge \mathbf{r}_v = (-3v^2, -3u^2, 9u^2v^2), \quad \mathbf{n} = \frac{1}{\sqrt{u^4 + v^4 + 9u^4v^4}}(-v^2, -u^2, 3u^2v^2).$$

由

$$\mathbf{r}_{uu} = (6u, 0, 0), \quad \mathbf{r}_{uv} = \mathbf{0}, \quad \mathbf{r}_{vv} = (0, 6v, 0),$$

有

$$\begin{aligned} L &= \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = -\frac{6uv^2}{\sqrt{u^4 + v^4 + 9u^4v^4}}, \\ M &= \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = 0, \\ N &= \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = -\frac{6u^2v}{\sqrt{u^4 + v^4 + 9u^4v^4}}. \end{aligned}$$

故曲面上的抛物点满足方程

$$LN - M^2 = \frac{36u^3v^3}{u^4 + v^4 + 9u^4v^4} = 0 \Leftrightarrow uv = 0 \Leftrightarrow u = 0 \text{ 或者 } v = 0.$$

当 $u = 0$ 时, 抛物点轨迹的参数方程为 $\mathbf{r}(0, v) = (0, v^3, v)$, 这是 yz -平面中的三次曲线: $y = z^3$.

当 $v = 0$ 时, 抛物点轨迹的参数方程为 $\mathbf{r}(u, 0) = (u^3, 0, u)$, 这是 xz -平面中的三次曲线: $x = z^3$.

25. 求曲面 $\mathbf{r}(u, v) = (a(u+v), b(u-v), 4uv)$ 的 Gauss 曲率、平均曲率、主曲率及对应的主方向.

解: 由

$$\mathbf{r}_u = (a, b, 4v), \quad \mathbf{r}_v = (a, -b, 4u),$$

有

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = a^2 + b^2 + 16v^2,$$

$$F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = a^2 - b^2 + 16uv,$$

$$G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = a^2 + b^2 + 16u^2.$$

又(记 $\Delta = 4(a^2 + b^2)(u^2 + v^2) + 8(b^2 - a^2)uv + a^2b^2$.)

$$\mathbf{r}_u \wedge \mathbf{r}_v = (4b(u+v), 4a(v-u), -2ab), \quad \mathbf{n} = \frac{1}{\sqrt{\Delta}}(2b(u+v), 2a(v-u), -ab),$$

$$\mathbf{r}_{uu} = \mathbf{0} = \mathbf{r}_{vv}, \quad \mathbf{r}_{uv} = (0, 0, 4),$$

有

$$L = N = 0, \quad M = -\frac{4ab}{\sqrt{\Delta}}.$$

从而, 平均曲率

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{ab(a^2 - b^2 + 16uv)}{\Delta^{\frac{3}{2}}},$$

Gauss 曲率

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{4a^2b^2}{\Delta^2}.$$

从而, 主曲率

$$k = H \pm \sqrt{H^2 - K} = \frac{ab(a^2 - b^2 + 16uv \pm \sqrt{(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)})}{\Delta^{\frac{3}{2}}}.$$

设切向量 $\lambda \mathbf{r}_u + \mu \mathbf{r}_v$ 是一个主方向, Weingarten 变换在自然基 $\{\mathbf{r}_u, \mathbf{r}_v\}$ 之下的系数矩阵为 A , 则

$$(\lambda \ \mu)(kI - A) = 0$$

即:

$$(\lambda \ \mu) \begin{pmatrix} \pm \frac{ab\sqrt{(a^2+b^2+16u^2)(a^2+b^2+16v^2)}}{\Delta^{\frac{3}{2}}} & \frac{ab(a^2+b^2+16v^2)}{\Delta^{\frac{3}{2}}} \\ \frac{ab(a^2+b^2+16u^2)}{\Delta^{\frac{3}{2}}} & \pm \frac{ab\sqrt{(a^2+b^2+16u^2)(a^2+b^2+16v^2)}}{\Delta^{\frac{3}{2}}} \end{pmatrix} = 0.$$

$$\Leftrightarrow \sqrt{a^2 + b^2 + 16v^2}\lambda \pm \sqrt{a^2 + b^2 + 16u^2}\mu = 0.$$

因此,

$$(\lambda \ \mu) = c(\sqrt{a^2 + b^2 + 16u^2} \mp \sqrt{a^2 + b^2 + 16v^2}),$$

其中 $c \in \mathbb{R}$ 为常数.

故对应的主方向

$$\mathbf{e} = c(\sqrt{a^2 + b^2 + 16u^2}\mathbf{r}_u \mp \sqrt{a^2 + b^2 + 16v^2}\mathbf{r}_v).$$

注: 主方向的另一种求法: 设切向量 $\mathbf{v} = \lambda \mathbf{r}_u + \mu \mathbf{r}_v$ 是点 $\mathbf{r}(u, v)$ 的一个主方向. 则有实数 k 使得 $\mathcal{W}(\mathbf{v}) = k\mathbf{v}$, 即: $-(\lambda \mathbf{n}_u + \mu \mathbf{n}_v) = k(\lambda \mathbf{r}_u + \mu \mathbf{r}_v)$. 这等价于(将上式分别与 $\mathbf{r}_u, \mathbf{r}_v$ 作内积得到)两个方程:

$$\begin{cases} L\lambda + M\mu = k(E\lambda + F\mu) \\ M\lambda + N\mu = k(F\lambda + G\mu) \end{cases} \quad (11)$$

而这两个方程等价于

$$\begin{vmatrix} L\lambda + M\mu & E\lambda + F\mu \\ M\lambda + N\mu & F\lambda + G\mu \end{vmatrix} = 0$$

$\Leftrightarrow$

$$(LF - ME)\lambda^2 + (LG - NE)\lambda\mu + (MG - NF)\mu^2 = 0$$

$\Leftrightarrow$

$$\begin{vmatrix} \mu^2 & -\lambda\mu & \lambda^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

故上面证明了: 切向量 $\mathbf{v} = \lambda\mathbf{r}_u + \mu\mathbf{r}_v$ 是点 $\mathbf{r}(u, v)$ 的一个主方向当且仅当

$$\begin{vmatrix} \mu^2 & -\lambda\mu & \lambda^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

对应的主曲率可以由式-(11) 得到:

$$k = \frac{L\lambda + M\mu}{E\lambda + F\mu} = \frac{M\lambda + N\mu}{F\lambda + G\mu}.$$

将上述方法应用到习题: 只需解方程

$$\begin{vmatrix} \mu^2 & -\lambda\mu & \lambda^2 \\ a^2 + b^2 + 16v^2 & a^2 - b^2 + 16uv & a^2 + b^2 + 16u^2 \\ 0 & -\frac{4ab}{\sqrt{\Delta}} & 0 \end{vmatrix} = 0.$$

$\Leftrightarrow$

$$(a^2 + b^2 + 16u^2)\mu^2 - (a^2 + b^2 + 16v^2)\lambda^2 = 0$$

$\Leftrightarrow$

$$\begin{pmatrix} \lambda & \mu \end{pmatrix} = c \left(\sqrt{a^2 + b^2 + 16u^2} \pm \sqrt{a^2 + b^2 + 16v^2} \right).$$

对应的主曲率

$$k = \frac{ab(a^2 - b^2 + 16uv \mp \sqrt{(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)})}{\Delta^{\frac{3}{2}}}.$$

26. 设 P 是曲面 S 上的一点. 证明: 当 P 不是脐点时, S 的主曲率 k_1, k_2 是 P 附近的光滑函数; 当 P 是脐点时, 主曲率是 P 附近的连续函数.

证明: 首先, 主曲率

$$k = H \pm \sqrt{H^2 - K}$$

是连续的. 注意到点 P 是脐点当且仅当 $H^2(P) - K(P) = 0$.

当 P 是脐点时, 主曲率在此点未必可微, 除非在其某个小邻域内所有点都是脐点.

当 P 是非脐点时, 由 H, K 的光滑性, 知在 P 点附近, 主曲率是光滑的.

27. 设曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 上没有抛物点, $\mathbf{n}$ 是 S 的法向量; 曲面 $\tilde{S}: \tilde{\mathbf{r}} = \mathbf{r}(u, v) + \lambda\mathbf{n}(u, v)$ (常数 λ 充分小) 称为 S 的平行曲面.

(1) 证明曲面 S 和 $\tilde{S}$ 在对应点的切平面平行;

(2) 可以选取 $\tilde{S}$ 的单位法向 $\tilde{\mathbf{n}}$, 使得 $\tilde{S}$ 的 Gauss 曲率和平均曲率分别为

$$\tilde{K} = \frac{K}{1 - 2\lambda H + \lambda^2 K}, \quad \tilde{H} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}.$$

证明: (1) 只需证明对曲面 S 上任意点 P 及其在曲面 $\tilde{S}$ 上的对应点 $\tilde{P}$ 的切空间相同, 即: $T_P S = T_{\tilde{P}} \tilde{S}$. 由 Weigarten 方程知, $\tilde{\mathbf{r}}_u = \mathbf{r}_u + \lambda \mathbf{n}_u$, $\tilde{\mathbf{r}}_v = \mathbf{r}_v + \lambda \mathbf{n}_v \in T_P S$. 而两个切空间都是二维的, 故 $T_P S = T_{\tilde{P}} \tilde{S}$.

(2) 方法一:

由 (1) 知, 曲面 $\tilde{S}$ 的单位法向量 $\tilde{\mathbf{n}} = \pm \mathbf{n}$.

(a): $\tilde{\mathbf{n}} = \mathbf{n}$. 设 $S, \tilde{S}$ 的 Weingarten 变换在坐标切向量下的系数矩阵分别为 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tilde{A}$. 由

$$\begin{pmatrix} \tilde{\mathbf{r}}_u \\ \tilde{\mathbf{r}}_v \end{pmatrix} = (I_2 - \lambda A) \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix},$$

有

$$\tilde{\mathcal{W}} \begin{pmatrix} \tilde{\mathbf{r}}_u \\ \tilde{\mathbf{r}}_v \end{pmatrix} = - \begin{pmatrix} \tilde{\mathbf{n}}_u \\ \tilde{\mathbf{n}}_v \end{pmatrix} = - \begin{pmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{pmatrix} = A \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix} = A(I_2 - \lambda A)^{-1} \begin{pmatrix} \tilde{\mathbf{r}}_u \\ \tilde{\mathbf{r}}_v \end{pmatrix}.$$

即:

$$\tilde{A} = A(I_2 - \lambda A)^{-1} = \frac{1}{1 - 2\lambda H + \lambda^2 K} \begin{pmatrix} a - \lambda K & b \\ c & d - \lambda K \end{pmatrix}.$$

因此,

$$\begin{aligned} \tilde{H} &= \frac{1}{2} \text{tr} \tilde{A} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}, \\ \tilde{K} &= \det \tilde{A} = \frac{K}{1 - 2\lambda H + \lambda^2 K}. \end{aligned}$$

(b): $\tilde{\mathbf{n}} = -\mathbf{n}$. 此时,

$$\tilde{A} = -A(I_2 - \lambda A)^{-1} = -\frac{1}{1 - 2\lambda H + \lambda^2 K} \begin{pmatrix} a - \lambda K & b \\ c & d - \lambda K \end{pmatrix}.$$

故

$$\begin{aligned} \tilde{H} &= \frac{1}{2} \text{tr} \tilde{A} = -\frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}, \\ \tilde{K} &= \det \tilde{A} = \frac{K}{1 - 2\lambda H + \lambda^2 K}. \end{aligned}$$

综上所述, 单位法向量 $\tilde{\mathbf{n}} = \mathbf{n}$ 即为所求.

方法二:

(a): $\tilde{\mathbf{n}} = \mathbf{n}$. 如方法一, 求得

$$\tilde{A} = A(I_2 - \lambda A)^{-1}.$$

曲面 S 的主曲率 k_1, k_2 是矩阵 A 的特征值, 故曲面 $\tilde{S}$ 的主曲率 $\tilde{k}_i$ 是矩阵 $\tilde{A}$ 的特征值:

$$\tilde{k}_i = \frac{k_i}{1 - \lambda k_i}, \quad i = 1, 2.$$

故

$$\begin{aligned} \tilde{H} &= \frac{1}{2}(\tilde{k}_1 + \tilde{k}_2) = \frac{\frac{1}{2}(k_1 + k_2) - \lambda k_1 k_2}{(1 - \lambda k_1)(1 - \lambda k_2)} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}, \\ \tilde{K} &= \tilde{k}_1 \tilde{k}_2 = \frac{k_1 k_2}{(1 - \lambda k_1)(1 - \lambda k_2)} = \frac{K}{1 - 2\lambda H + \lambda^2 K}. \end{aligned}$$

(b): $\tilde{\mathbf{n}} = -\mathbf{n}$. 此时,

$$\tilde{A} = -A(I_2 - \lambda A)^{-1}.$$

故

$$\tilde{k}_i = -\frac{k_i}{1 - \lambda k_i}, \quad i = 1, 2.$$

从而,

$$\begin{aligned} \tilde{H} &= \frac{1}{2}(\tilde{k}_1 + \tilde{k}_2) = -\frac{\frac{1}{2}(k_1 + k_2) - \lambda k_1 k_2}{(1 - \lambda k_1)(1 - \lambda k_2)} = -\frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}, \\ \tilde{K} &= \tilde{k}_1 \tilde{k}_2 = \frac{k_1 k_2}{(1 - \lambda k_1)(1 - \lambda k_2)} = \frac{K}{1 - 2\lambda H + \lambda^2 K}. \end{aligned}$$

综上所述, 单位法向量 $\tilde{\mathbf{n}} = \mathbf{n}$ 即为所求.

注: 之所以要求曲面 S 没有抛物点, 是因为要取到充分小的常数 λ 使得曲面 $\tilde{S}$ 有定义. 具体地, 由

$$\begin{pmatrix} \tilde{\mathbf{r}}_u \\ \tilde{\mathbf{r}}_v \end{pmatrix} = (I_2 - \lambda A) \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix}$$

知, 曲面 $\tilde{S}$ 有定义当且仅当矩阵 $I_2 - \lambda A$ 可逆当且仅当

$$\det(I_2 - \lambda A) = 1 - 2H\lambda + \lambda^2 K \neq 0$$

当且仅当 $\lambda k_1 = \lambda(H + \sqrt{H^2 - K}) \neq 1$ 且 $\lambda k_2 = \lambda(H - \sqrt{H^2 - K}) \neq 1$.

曲面 S 没有抛物点等价于 K 恒不为 0 等价于两个主曲率均恒不为 0. 这样在局部上, 总可以取充分小的常数 λ 使得它取值于 $\frac{1}{k_1}$ 和 $\frac{1}{k_2}$ 之间. 从而, $1 - 2H\lambda + \lambda^2 K \neq 0$, 即: 曲面 $\tilde{S}$ 有定义.

28. 曲面 S 上的一条曲线 C 称为曲率线, 如果 C 在每点的切向量都是曲面 S 在该点的一个主方向. 证明: 曲线 $C: \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ 是曲率线当且仅当沿着 C , $\frac{d\mathbf{n}(t)}{dt}$ 与 $\frac{d\mathbf{r}(t)}{dt}$ 平行.

证明: 曲线 $C: \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ 是曲率线当且仅当

$$-\mathbf{n}'(t) = -(\mathbf{n}_u(t)u'(t) + \mathbf{n}_v(t)v'(t)) = \mathcal{W}(\mathbf{r}'(t)) = k\mathbf{r}'(t),$$

对某个 $k \in \mathbb{R}$. 而这等价于 $\mathbf{r}'(t) // \mathbf{n}'(t)$.

29. 设 $\mathbf{r} = \mathbf{r}(u, v)$ 是一个无脐点的曲面的参数表示. 证明: 曲面 S 的参数曲线 $u = \text{常数}$ 和 $v = \text{常数}$ 是曲率线的充要条件是 $F = M = 0$.

证明: 设曲面 S 的参数曲线是曲率线, 则

$$-\mathbf{n}_u = \mathcal{W}(\mathbf{r}_u) = k_1 \mathbf{r}_u,$$

$$-\mathbf{n}_v = \mathcal{W}(\mathbf{r}_v) = k_2 \mathbf{r}_v,$$

其中主曲率 k_1 和 k_2 不相等(这是因为曲面无脐点). 那么对应的主方向必然是垂直的, 即:

$$F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0.$$

从而,

$$M = -\langle \mathbf{r}_u, \mathbf{n}_v \rangle = k_2 \langle \mathbf{r}_u, \mathbf{r}_v \rangle = k_2 F = 0.$$

反过来, 设 $F = M = 0$. 故 $\mathbf{r}_u \perp \mathbf{r}_v$ 且 $\mathbf{r}_u \perp \mathbf{n}_v$. 由于 $\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}_v$ 共面, $\mathbf{n}_v = -k_2 \mathbf{r}_v$ 对某个 $k_2 \in \mathbb{R}$. 从而,

$$\mathcal{W}(\mathbf{r}_v) = -\mathbf{n}_v = k_2 \mathbf{r}_v.$$

即: $\mathbf{r}_v$ 是主方向. 同理, $\mathbf{r}_u$ 也是主方向. 因此, 曲面 S 的参数曲线是曲率线.

注: 事实上, 上面也证明了: 对任意曲面 S 的参数表达式 $\mathbf{r} = \mathbf{r}(u, v)$, 坐标切向量是相互正交的主方向当且仅当 $F = M = 0$. 此时, 主曲率 $k_1 = \frac{L}{E}$, $k_2 = \frac{N}{G}$.

30. 求曲面 $F(x, y, z) = 0$ 的曲率线所满足的微分方程.

解: 设曲线 $C: \mathbf{r}(t) = (x(t), y(t), z(t))$ 在曲面 $S: F(x, y, z) = 0$ 上. 则

$$F(x(t), y(t), z(t)) = 0.$$

故

$$F_x(x(t), y(t), z(t))x'(t) + F_y(x(t), y(t), z(t))y'(t) + F_z(x(t), y(t), z(t))z'(t) = 0,$$

即:

$$F_x dx + F_y dy + F_z dz = 0.$$

从而,

$$\mathbf{n} = \pm \frac{1}{\sqrt{F_x^2 + F_y^2 + F_z^2}} (F_x, F_y, F_z).$$

曲线 C 是曲率线当且仅当沿着 C , 存在某个实函数 k 使得

$$-d\mathbf{n} = \mathcal{W}(d\mathbf{r}) = k d\mathbf{r}$$

$\Leftrightarrow$

$$d\mathbf{r} // d\mathbf{n}$$

$\Leftrightarrow$

$$d\mathbf{r} \wedge d\mathbf{n} = \mathbf{0}.$$

现在

$$d\mathbf{r} = (dx, dy, dz),$$

满足

$$F_x dx + F_y dy + F_z dz = 0.$$

而

$$\begin{aligned} d\mathbf{n} = & \pm \frac{1}{(F_x^2 + F_y^2 + F_z^2)^{\frac{3}{2}}} ((F_y^2 + F_z^2 - F_x(F_y + F_z))(F_{xx}dx + F_{xy}dy + F_{xz}dz), \\ & (F_x^2 + F_z^2 - F_y(F_x + F_z))(F_{xy}dx + F_{yy}dy + F_{yz}dz), \\ & (F_x^2 + F_y^2 - F_z(F_x + F_y))(F_{xz}dx + F_{yz}dy + F_{zz}dz)). \end{aligned}$$

记

$$\begin{aligned} \phi &= (F_y^2 + F_z^2 - F_x(F_y + F_z))(F_{xx}dx + F_{xy}dy + F_{xz}dz), \\ \varphi &= (F_x^2 + F_z^2 - F_y(F_x + F_z))(F_{xy}dx + F_{yy}dy + F_{yz}dz), \\ \psi &= (F_x^2 + F_y^2 - F_z(F_x + F_y))(F_{xz}dx + F_{yz}dy + F_{zz}dz). \end{aligned}$$

则

$$d\mathbf{r} \wedge d\mathbf{n} = \mathbf{0}$$

$\Leftrightarrow$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & dy & dz \\ \phi & \varphi & \psi \end{vmatrix} = \mathbf{0}$$

$\Leftrightarrow$

$$\begin{cases} dy\psi - dz\varphi = 0 \\ dx\psi - dz\phi = 0 \\ dx\varphi - dy\phi = 0. \end{cases} \quad (12)$$

故曲面 S 上的曲率线满足的微分方程为

$$\begin{cases} F_x dx + F_y dy + F_z dz = 0 \\ dy\psi - dz\varphi = 0 \\ dx\psi - dz\phi = 0 \\ dx\varphi - dy\phi = 0. \end{cases} \quad (13)$$

注: 设曲面 S 的参数表示式为 $\mathbf{r} = \mathbf{r}(u, v)$. 由习题 25 后的注, 曲面 S 上的曲线 $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ 是曲率线当且仅当

$$\begin{vmatrix} u'(t)^2 & -u'(t)v'(t) & v'(t)^2 \\ E(u(t), v(t)) & F(u(t), v(t)) & G(u(t), v(t)) \\ L(u(t), v(t)) & M(u(t), v(t)) & N(u(t), v(t)) \end{vmatrix} = 0$$

$\Leftrightarrow$

$$\begin{vmatrix} \left(\frac{du(t)}{dt}\right)^2 & -\frac{du(t)}{dt}\frac{dv(t)}{dt} & \left(\frac{dv(t)}{dt}\right)^2 \\ E(u(t), v(t)) & F(u(t), v(t)) & G(u(t), v(t)) \\ L(u(t), v(t)) & M(u(t), v(t)) & N(u(t), v(t)) \end{vmatrix} = 0$$

$\Leftrightarrow$

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

这就是曲面 S 上曲率线满足的微分方程. 反过来, 其积分曲线是曲率线.

31. 曲面 S 上的一个切方向是渐进方向, 如果沿此方向的法曲率为 0; S 上的一条曲线 C 称为渐进线, 如果它的每个切方向都是渐进方向. 证明曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 的参数曲线是渐近线当且仅当 $L = N = 0$.

证明: 曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 的参数曲线是渐近线当且仅当

$$\mathcal{W}(\mathbf{r}_u) = \frac{L}{E} = 0 \quad \text{且} \quad \mathcal{W}(\mathbf{r}_v) = \frac{N}{G} = 0$$

当且仅当

$$L = N = 0.$$

注: 由定义, 切向量 $\lambda\mathbf{r}_u + \mu\mathbf{r}_v$ 是渐进方向当且仅当

$$L\lambda^2 + 2M\lambda\mu + N\mu^2 = 0.$$

从而, 渐进曲线满足的微分方程为

$$Ldu^2 + 2Mdudv + Ndv^2 = 0.$$

反过来, 此微分方程的积分曲线是渐近线.

在点 $\mathbf{r}(u, v)$, 有两个线性无关的渐进方向当且仅当

$$LN - M^2 < 0,$$

即: 点 $\mathbf{r}(u, v)$ 是双曲点. 故由习题 12 的证明, 在双曲点附近, 存在渐进曲线网. 反过来, 曲面在一点附近存在渐进曲线网, 则此点必为双曲点.

32. 证明: 若曲面的切平面过定点, 则该曲面是锥面.

证明: 方法一:

设曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 的切平面过定点 P_0 , 其位置向量为 $\mathbf{p}_0$. 则

$$\mathbf{r}(u, v) - \mathbf{p}_0 = \lambda(u, v)\mathbf{r}_u + \mu(u, v)\mathbf{r}_v,$$

其中 $\lambda(u, v), \mu(u, v)$ 是光滑函数. 从而,

$$\mathbf{r}_u = \lambda_u \mathbf{r}_u + \lambda \mathbf{r}_{uu} + \mu_u \mathbf{r}_v + \mu \mathbf{r}_{uv}, \quad \mathbf{r}_v = \lambda_v \mathbf{r}_u + \lambda \mathbf{r}_{uv} + \mu_v \mathbf{r}_v + \mu \mathbf{r}_{vv}.$$

将以上两式与 $\mathbf{n}$ 作内积, 有

$$\lambda L + \mu M = 0,$$

$$\lambda M + \mu N = 0.$$

故

$$\lambda(LN - M^2) = 0,$$

$$\mu(LN - M^2) = 0.$$

由于 $\lambda(u, v), \mu(u, v)$ 只在一点同时为 0, 故 $LN - M^2 = 0$. 从而, Gauss 曲率 $K = \frac{LN - M^2}{EG - F^2} = 0$.

设 S 上的点 P 是非脐点, 则在它的一个小邻域内, S 无脐点. 由习题 12, 对应于两个主方向量场, 在更小的邻域内, S 有正交参数, 仍记为 (u, v) . (对应的参数曲线是正交曲率线) 而由 $K = 0$, 此小邻域内每点都是严格抛物点(非平点), 只沿一个方向法曲率为 0. 故其中一族参数曲线是曲率线且是渐近线. 而沿着方向 $\mathbf{r}(u, v) - \mathbf{p}_0$, 法曲率

$$k_n(\mathbf{r}(u, v) - \mathbf{p}_0) = \frac{L\lambda^2 + 2M\lambda\mu + N\mu^2}{E\lambda^2 + 2F\lambda\mu + G\mu^2} = 0.$$

因此, 这族曲率渐近线的切方向都过同一定点 P_0 . 由习题二 9 (1), 它们必是一束直线.

现在设 S 上点 P 是脐点, 则它是平点. 若存在 P 的一个邻域, S 上每点都是平点. 则 S 在此邻域内是平面的一部分. 若 P 不存在这样的邻域, 则在 P 的附近, 脐点的轨迹至多是一些曲线, 不能决定曲面的形状.

综上所述, 曲面 S 上每点都在曲面上的一条直线上且所有这些直线过定点, 即: S 是锥面.

方法二: (初等的几何证明)

设曲面 S 的所有切平面过定点 P_0 . 取曲面上任意点 $P \neq P_0$. 设点 P_0 与过点 P 法线张成的平面为 Π , 而曲面 S 与平面 Π 的相交曲线为 C . 对于 C 上任意一点 Q , 直线 $\overline{P_0Q}$ 在平面 Π 中. 而由假设, S 的切平面都过 P_0 , 故在 Q 点附近, 曲线 C 只在直线 $\overline{P_0Q}$ 的一侧. 类似于习题 5, 点 Q 是平面 Π 中曲线 C 的高度函数的极小值点, 故直线 $\overline{P_0Q}$ 是曲线 C 在点 Q 的切线. 由习题二 9 (1), 曲线 C 必是直线. 因此, S 由过定点的直线构成, 是锥面.

33. 证明: 直纹面是可展曲面当且仅当沿着直母线, 曲面的切平面不变.

证明: 设直纹面 S 的参数式为 $\mathbf{r}(u, v) = \mathbf{a}(u) + v\mathbf{b}(u)$.

沿着直母线, 曲面的切平面不变 $\Leftrightarrow$ 对于任意 u , 及 $v_1 \neq v_2$, $\mathbf{n}(u, v_1) // \mathbf{n}(u, v_2)$

$\Leftrightarrow$

$$\mathbf{r}_u(u, v_1) \wedge \mathbf{r}_v(u, v_1) // \mathbf{r}_u(u, v_2) \wedge \mathbf{r}_v(u, v_2)$$

$\Leftrightarrow$

$$0 = (\mathbf{r}_u(u, v_1) \wedge \mathbf{r}_v(u, v_1)) \wedge (\mathbf{r}_u(u, v_2) \wedge \mathbf{r}_v(u, v_2)) = ((\mathbf{a}' + v_1 \mathbf{b}') \wedge \mathbf{b}) \wedge ((\mathbf{a}' + v_2 \mathbf{b}') \wedge \mathbf{b})$$

$$\begin{aligned}
&= \langle \mathbf{a}' + v_1 \mathbf{b}', (\mathbf{a}' + v_2 \mathbf{b}') \wedge \mathbf{b} \rangle \mathbf{b} - \langle \mathbf{b}, (\mathbf{a}' + v_2 \mathbf{b}') \wedge \mathbf{b} \rangle (\mathbf{a}' + v_1 \mathbf{b}') \\
&= \langle \mathbf{a}', v_2 \mathbf{b}' \wedge \mathbf{b} \rangle + \langle v_1 \mathbf{b}', \mathbf{a}' \wedge \mathbf{b} \rangle = (v_1 - v_2) \langle \mathbf{a}', \mathbf{b}, \mathbf{b}' \rangle
\end{aligned}$$

$\Leftrightarrow$

$$(\mathbf{a}', \mathbf{b}, \mathbf{b}') = 0$$

$\Leftrightarrow$ 曲面 S 是可展曲面(性质 6.1).

34. 求曲面 $\mathbf{r}(u, v) = (3u + 3uv^2 - u^3, v^3 - 3v - 3u^2v, 3(u^2 - v^2))$ 的平均曲率和 Gauss 映射的像集.

解: 由

$$\mathbf{r}_u = (3 - 3v^2 - 3u^2, -6uv, 6u), \quad \mathbf{r}_v = (6uv, 3v^2 - 3 - 3u^2, -6v),$$

有

$$E = G = 9(1 + u^2 + v^2)^2, \quad F = 0.$$

由

$$\mathbf{r}_u \wedge \mathbf{r}_v = (18u(1 + u^2 + v^2), 18v(1 + u^2 + v^2), 9(u^2 + v^2 - 1)(1 + u^2 + v^2)),$$

有

$$\mathbf{n} = \frac{1}{1 + u^2 + v^2} (2u, 2v, u^2 + v^2 - 1).$$

而

$$\mathbf{r}_{uu} = (-6u, -6v, 6), \quad \mathbf{r}_{uv} = (6v, -6u, 0), \quad \mathbf{r}_{vv} = (6u, 6v, -6),$$

故

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = -6, \quad M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = 0, \quad N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = 6.$$

因此,

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0, \quad G = \frac{LN - M^2}{EG - F^2} = -\frac{4}{9(1 + u^2 + v^2)^4}.$$

Gauss 映射 $g: S \rightarrow S^2$ 的像集等于映射 $\mathbf{r}(u, v): E^2 \rightarrow S$ 与 g 的合成的像集, 即是映射 $\mathbf{n}(u, v): E^2 \rightarrow S^2$ 的像集. 而这正是单位球极投影坐标参数式的像集 $S^2 - \{(0, 0, 1)\}$.

35. 证明: 形如 $z = f(x) + g(y)$ 的极小曲面, 若非平面, 则除相差一个常数外, 它可以写成

$$z = \frac{1}{a} \log \frac{\cos ay}{\cos ax}.$$

此曲面称为 Scherk 曲面.

证明: 设方程 $z = f(x) + g(y)$ 定义了一个极小曲面 S . 显然, 它有参数表达式

$$\mathbf{r}(u, v) = (u, v, f(u) + g(v)).$$

直接计算, 有

$$\mathbf{r}_u = (1, 0, f'(u)), \quad \mathbf{r}_v = (0, 1, g'(v)).$$

故

$$E = 1 + f'(u)^2, \quad F = f'(u)g'(v), \quad G = 1 + g'(v)^2.$$

由

$$\mathbf{r}_u \wedge \mathbf{r}_v = (-f'(u), -g'(v), 1),$$

知

$$\mathbf{n} = \frac{1}{\sqrt{1 + f'(u)^2 + g'(v)^2}} (-f'(u), -g'(v), 1).$$

而

$$\mathbf{r}_{uu} = (0, 0, f''(u)), \quad \mathbf{r}_{uv} = \mathbf{0}, \quad \mathbf{r}_{vv} = (0, 0, g''(v)),$$

故

$$L = \frac{f''(u)}{\sqrt{1 + f'(u)^2 + g'(v)^2}}, \quad M = 0, \quad N = \frac{g''(v)}{\sqrt{1 + f'(u)^2 + g'(v)^2}}.$$

由于曲面是极小的,

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{f''(u)(1 + g'(v)^2) + g''(v)(1 + f'(u)^2)}{2(1 + f'(u)^2 + g'(v)^2)^{\frac{3}{2}}} = 0.$$

这等价于

$$f''(u)(1 + g'(v)^2) + g''(v)(1 + f'(u)^2) = 0.$$

继而, 等价于

$$\frac{f''(u)}{1 + f'(u)^2} = -\frac{g''(v)}{1 + g'(v)^2}.$$

由于上面等式两边是关于不同变量的函数, 故它们只能为同一常数, 记为 a . 若 S 不是平面, 则必有 $a \neq 0$. 否则, 由 $f''(u) = 0 = g''(v)$, 有 $f(u) = ku + m$, $g(v) = lv + n$, 其中. 从而, S 是平面.

现在

$$d(\arctan f'(u)) = d(au), \quad d(\arctan g'(v)) = d(-av).$$

故

$$\arctan f'(u) = au + b, \quad \arctan g'(v) = -av - c,$$

其中 b, c 为常数. 从而,

$$f(u) = -\frac{1}{a} \log \cos(au + b) + d, \quad g(v) = \frac{1}{a} \log \cos(av + c) + e,$$

其中 d, e 为常数. 所以,

$$z = f(x) + g(y) = \frac{1}{a} \log \frac{\cos(ay + c)}{\cos(ax + b)} + d + e.$$

36. 证明: 正螺旋面 $\mathbf{r}(u, v) = (u \cos v, u \sin v, bv)$ 是极小曲面; 并证明: 直纹极小曲面是平面或者正螺旋面.

证明: 直接计算, 有

$$\mathbf{r}_u = (\cos v, \sin v, 0), \quad \mathbf{r}_v = (-u \cos v, u \sin v, b), \quad \mathbf{n} = \frac{1}{\sqrt{u^2 + b^2}}(b \sin v, b \cos v, u).$$

故

$$E = 1, \quad F = 0, \quad G = u^2 + b^2.$$

又

$$\mathbf{r}_{uu} = \mathbf{0}, \quad \mathbf{r}_{uv} = (-\sin v, \cos v, 0), \quad \mathbf{r}_{vv} = (-u \cos v, -u \sin v, 0),$$

故

$$L = 0, \quad M = -\frac{b}{\sqrt{u^2 + b^2}}, \quad N = 0.$$

从而, 平均曲率

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0.$$

即: 正螺旋面是极小曲面.

设极小直纹面 S 的参数表示为 $\mathbf{r}(u, v) = \mathbf{a}(u) + v\mathbf{c}(u)$. 则

$$\mathbf{r}_u = \mathbf{a}' + v\mathbf{c}', \quad \mathbf{r}_v = \mathbf{c}, \quad \mathbf{r}_u \wedge \mathbf{r}_v = \mathbf{a}' \wedge \mathbf{c} + v\mathbf{c}' \wedge \mathbf{c}.$$

故

$$E = \langle \mathbf{a}', \mathbf{a}' \rangle + v\langle \mathbf{a}', \mathbf{c}' \rangle + v^2\langle \mathbf{c}', \mathbf{c}' \rangle, \quad F = \mathbf{a}' \wedge \mathbf{c} + v\mathbf{c}' \wedge \mathbf{c}, \quad G = \langle \mathbf{c}, \mathbf{c} \rangle.$$

为取得 (u, v) 是正交参数, 即: $F = 0$, 可以假设 $|\mathbf{c}(u)| = 1$; 然后, 经过参数变换 $\tilde{u} = u$, $\tilde{v} = v + \int_0^u \langle \mathbf{a}'(t), \mathbf{c}(t) \rangle dt$, 可以设 $\langle \mathbf{a}'(u), \mathbf{c}(u) \rangle = 0$. 此时, $F = 0$ 且 $G = 1$. 记 $\Delta = |\mathbf{r}_u \wedge \mathbf{r}_v|$.

由

$$\mathbf{r}_{uu} = \mathbf{a}'' + v\mathbf{c}'', \quad \mathbf{r}_{uv} = \mathbf{c}', \quad \mathbf{r}_{vv} = \mathbf{0},$$

有

$$L = \frac{1}{\Delta}[(\mathbf{a}'', \mathbf{a}', \mathbf{c}) + v((\mathbf{a}', \mathbf{c}, \mathbf{c}'') + (\mathbf{a}'', \mathbf{c}', \mathbf{c})) + v^2(\mathbf{c}'', \mathbf{c}', \mathbf{c})], \quad M = \frac{1}{\Delta}(\mathbf{a}', \mathbf{c}, \mathbf{c}'), \quad N = 0.$$

曲面 S 是极小的 $\Leftrightarrow H = 0 \Leftrightarrow$

$$0 = \Delta(LG - 2MF + NE) = (\mathbf{a}'', \mathbf{a}', \mathbf{c}) + v((\mathbf{a}', \mathbf{c}, \mathbf{c}'') + (\mathbf{a}'', \mathbf{c}', \mathbf{c})) + v^2(\mathbf{c}'', \mathbf{c}', \mathbf{c}).$$

$\Leftrightarrow$

$$\begin{cases} (\mathbf{a}'', \mathbf{a}', \mathbf{c}) = 0 \\ (\mathbf{a}', \mathbf{c}, \mathbf{c}'') + (\mathbf{a}'', \mathbf{c}', \mathbf{c}) = 0 \\ (\mathbf{c}'', \mathbf{c}', \mathbf{c}) = 0. \end{cases} \quad (14)$$

由式-(14) 中第三式, 知 $\mathbf{c}(u)$ 在某个平面上. 而由假设, $\mathbf{c}(u)$ 是一条单位球面曲线(注意 $\mathbf{c}(u)$ 不能为常向量), 故 $\mathbf{c}(u)$ 是一个单位圆.

若 $\mathbf{a}(u)$ 为常向量, 则 S 是平面. 现在假设 $\mathbf{a}(u)$ 不为常向量, 即: 它是一条曲线. 可以设 u 是曲线 $\mathbf{a}(u)$ 的弧长参数, 其 Frenet 标架为 $\{\mathbf{a}(u); \mathbf{t}(u), \mathbf{n}(u), \mathbf{b}(u)\}$, 其曲率为 $\kappa(u)$, 挠率为 $\tau(u)$. 由式-(14) 中第一式, 有

$$0 = (\mathbf{a}'', \mathbf{a}', \mathbf{c}) = \kappa(\mathbf{n}, \mathbf{t}, \mathbf{c}) = -\kappa\langle \mathbf{b}, \mathbf{c} \rangle.$$

若 $\kappa = 0$, 则 $\mathbf{a}(u)$ 是直线. 而 $\mathbf{c}(u)$ 是一个单位圆. 可以设

$$\mathbf{a}(u) = (0, 0, bu).$$

由假设 $\mathbf{t} \perp \mathbf{c}(u)$, 故

$$\mathbf{c}(u) = (\cos u, \sin u, 0).$$

从而, 曲面 S 的参数表达式为

$$\mathbf{r}(u, v) = \mathbf{a}(u) + v\mathbf{c}(u) = (v \cos u, v \sin u, bu).$$

即: 曲面 S 为正螺旋面.

若 κ 不恒为 0, 只需考虑 $\kappa \neq 0$ 的部分, 则 $\langle \mathbf{b}, \mathbf{c} \rangle = 0$. 而由假设, $\langle \mathbf{t}, \mathbf{c} \rangle = 0$. 故 $\mathbf{c} = \pm \mathbf{n}$. 从而, 可以设 $\mathbf{c} = \mathbf{n}$. 由式-(14) 中第二式, 有

$$0 = (\mathbf{a}', \mathbf{c}, \mathbf{c}'') + (\mathbf{a}'', \mathbf{c}', \mathbf{c}) = (\mathbf{t}, \mathbf{n}, \ddot{\mathbf{n}}) + (\dot{\mathbf{t}}, \dot{\mathbf{n}}, \mathbf{n}) = \dot{\tau}.$$

故 τ 是常数.

若 $\tau = 0$, 则 $\mathbf{a}(u)$ 为平面曲线. 而 $\mathbf{c} = \mathbf{n}$ 是其(主)法向量, 故 S 是平面.

若 $\tau \neq 0$, 则由式-(14) 中第三式, 有

$$0 = (\mathbf{c}'', \mathbf{c}', \mathbf{c}) = (\ddot{\mathbf{n}}, \dot{\mathbf{n}}, \mathbf{n}) = \dot{\kappa}\tau.$$

因此, $\dot{\kappa} = 0$, 即: κ 为常数. 故 $\mathbf{a}(u)$ 是圆柱螺旋线. 可以设

$$\mathbf{a}(u) = \left(\frac{\kappa}{\kappa^2 + \tau^2} \cos(\sqrt{\kappa^2 + \tau^2}u), \frac{\kappa}{\kappa^2 + \tau^2} \sin(\sqrt{\kappa^2 + \tau^2}u), \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}u \right)$$

则

$$\mathbf{c}(u) = \mathbf{n}(u) = (-\cos(\sqrt{\kappa^2 + \tau^2}u), -\sin(\sqrt{\kappa^2 + \tau^2}u), 0).$$

故

$$\begin{aligned} \mathbf{r}(u, v) = \mathbf{a}(u) + v\mathbf{c}(u) = & \left(\left(\frac{\kappa}{\kappa^2 + \tau^2} - v \right) \cos(\sqrt{\kappa^2 + \tau^2}u), \right. \\ & \left. \left(\frac{\kappa}{\kappa^2 + \tau^2} - v \right) \sin(\sqrt{\kappa^2 + \tau^2}u), \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}u \right). \end{aligned}$$

作参数变换 $\tilde{v} = \sqrt{\kappa^2 + \tau^2}u$, $\tilde{u} = \frac{\kappa}{\kappa^2 + \tau^2} - v$, 则曲面 S 的参数表达式变为

$$\mathbf{r}(\tilde{u}, \tilde{v}) = (\tilde{u} \cos \tilde{v}, \tilde{u} \sin \tilde{v}, \frac{\tau}{\kappa^2 + \tau^2} \tilde{v}).$$

故曲面 S 是正螺旋面.

习题四

1. (1) $g^{\alpha\beta}g_{\alpha\beta} = \delta^\alpha_\alpha = 2$.

(2) $\frac{\partial \ln \sqrt{g}}{\partial u^\alpha} = \frac{1}{\sqrt{g}} \cdot \frac{\partial g}{\partial u^\alpha} = \frac{1}{2g} \frac{\partial g}{\partial u^\alpha} = \frac{1}{2g} (g_{11} \frac{\partial g_{22}}{\partial u^\alpha} + g_{22} \frac{\partial g_{11}}{\partial u^\alpha} - 2g_{12} \frac{\partial g_{12}}{\partial u^\alpha})$
 $(g = g_{11}g_{22}g_{12}^2 \Rightarrow \frac{\partial g}{\partial u^\alpha} = g_{11} \frac{\partial g_{22}}{\partial u^\alpha} + g_{22} \frac{\partial g_{11}}{\partial u^\alpha} - 2g_{12} \frac{\partial g_{12}}{\partial u^\alpha}) = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial u^\alpha} + g^{12} \frac{\partial g_{12}}{\partial u^\alpha} + \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial u^\alpha}$

$$\begin{aligned} \Gamma_{1\alpha}^1 + \Gamma_{2\alpha}^2 &= \frac{1}{2} g^{1\beta} \left(\frac{\partial g_{1\beta}}{\partial u^\alpha} + \frac{\partial g_{\beta 1}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^1} \right) + \frac{1}{2} g^{2\beta} \left(\frac{\partial g_{2\beta}}{\partial u^\alpha} + \frac{\partial g_{\beta 2}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^2} \right) \\ &= \frac{1}{2} (g^{11} \frac{\partial g_{11}}{\partial u^\alpha} + g^{11} \frac{\partial g_{11}}{\partial u^\alpha} - g^{11} \frac{\partial g_{11}}{\partial u^1}) + \frac{1}{2} (g^{12} \frac{\partial g_{12}}{\partial u^\alpha} + g^{12} \frac{\partial g_{12}}{\partial u^\alpha} - g^{12} \frac{\partial g_{12}}{\partial u^2}) \\ &\quad + \frac{1}{2} (g^{12} \frac{\partial g_{12}}{\partial u^\alpha} + g^{12} \frac{\partial g_{12}}{\partial u^\alpha} - g^{12} \frac{\partial g_{12}}{\partial u^1}) + \frac{1}{2} (g^{22} \frac{\partial g_{22}}{\partial u^\alpha} + g^{22} \frac{\partial g_{22}}{\partial u^\alpha} - g^{22} \frac{\partial g_{22}}{\partial u^2}) \\ &= \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial u^\alpha} + g^{12} \frac{\partial g_{12}}{\partial u^\alpha} + \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial u^\alpha} \\ &= \frac{\partial \ln \sqrt{g}}{\partial u^\alpha} \end{aligned}$$

2. (1) $\tilde{r} = \tilde{r}(\tilde{u}^1, \tilde{u}^2) = r(u^1(\tilde{u}^1, \tilde{u}^2), u^2(\tilde{u}^1, \tilde{u}^2))$. $\tilde{r}_i = r_\alpha \cdot \frac{\partial u^\alpha}{\partial \tilde{u}^i} = r_\alpha \cdot a_i^\alpha$.

$\tilde{r}_{ij} = r_{\alpha\beta} \cdot a_j^\beta \cdot a_i^\alpha + r_\alpha \cdot a_{ij}^\alpha$ ($a_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial \tilde{u}^i \partial \tilde{u}^j}$)

$\tilde{g}_{ij} = \langle \tilde{r}_i, \tilde{r}_j \rangle = \langle r_\alpha \cdot a_i^\alpha, r_\beta \cdot a_j^\beta \rangle = g_{\alpha\beta} \cdot a_i^\alpha \cdot a_j^\beta$.

$\tilde{r}_1 \wedge \tilde{r}_2 = \det(a_i^\alpha) r_1 \wedge r_2$. $\tilde{n} = \frac{\tilde{r}_1 \wedge \tilde{r}_2}{|\tilde{r}_1 \wedge \tilde{r}_2|} = \text{sgn}(\det(a_i^\alpha)) n$.

$\tilde{b}_{ij} = \langle \tilde{r}_{ij}, \tilde{n} \rangle = \langle r_{\alpha\beta} a_i^\alpha \cdot a_j^\beta + r_\alpha \cdot a_{ij}^\alpha, \text{sgn}(\det(a_i^\alpha)) n \rangle$
 $= \text{sgn}(\det(a_i^\alpha)) b_{\alpha\beta} a_i^\alpha \cdot a_j^\beta$.

3. $\frac{1}{2} b_{\alpha\beta} \cdot g^{\beta\alpha} = \frac{1}{2} b_{\alpha}^{\alpha} = H$, (b_{α}^{β}) 是 Weingarten 变换矩阵.

4. $(g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, $(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$. $T_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\delta} \left(\frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}} \right)$

$\gamma=1$, $T_{\alpha\beta}^1 = \frac{1}{2} g^{11} \left(\frac{\partial g_{\alpha 1}}{\partial u^{\beta}} + \frac{\partial g_{\beta 1}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^1} \right)$

$(\alpha, \beta) = (2, 2)$, $T_{22}^1 = -\frac{1}{2} \frac{\partial g_{22}}{\partial u^1} = r$

$(\alpha, \beta) \neq (2, 2)$, $T_{\alpha\beta}^1 = 0$

$\gamma=2$, $T_{\alpha\beta}^2 = \frac{1}{2} g^{12} \left(\frac{\partial g_{2\alpha}}{\partial u^{\beta}} + \frac{\partial g_{2\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^2} \right)$

$(\alpha, \beta) = (1, 2)$, $T_{12}^2 = T_{21}^2 = \frac{1}{2} g^{22} \cdot \frac{\partial g_{12}}{\partial u^1} = \frac{1}{2} \cdot \frac{1}{r^2} \cdot 2r = \frac{1}{r}$

$(\alpha, \beta) \neq (1, 2)$, $T_{\alpha\beta}^2 = 0$.

5. $I = (1+f_x^2)dx^2 + 2f_x f_y dx dy + (1+f_y^2)dy^2$

$(g_{\alpha\beta}) = \begin{pmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{pmatrix}$

$(g^{\alpha\beta}) = \frac{1}{1+f_x^2+f_y^2} \begin{pmatrix} 1+f_y^2 & -f_x f_y \\ -f_x f_y & 1+f_x^2 \end{pmatrix}$

$T_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\delta} \left(\frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}} \right)$

$T_{11}^1 = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial u^1} + g^{12} \frac{\partial g_{12}}{\partial u^1} - \frac{1}{2} g^{12} \frac{\partial g_{11}}{\partial u^2}$

$= \frac{1}{2} \frac{1+f_y^2}{1+f_x^2+f_y^2} \frac{2f_y f_{xx} (1+f_x^2+f_y^2) - 2(f_x f_{xx} + f_y f_{xy})(1+f_y^2)}{(1+f_x^2+f_y^2)^2} + \left(\frac{f_x f_y}{1+f_x^2+f_y^2} \right) \frac{(f_{xx} f_y + f_x f_{xy})(1+f_x^2+f_y^2) - (2f_x f_{xx} + f_y f_{xy}) f_x f_y}{(1+f_x^2+f_y^2)^2}$

$- \frac{1}{2} \left(\frac{f_x f_y}{1+f_x^2+f_y^2} \right) \cdot \frac{2f_x f_{xy} (1+f_x^2+f_y^2) - 2(f_x f_{xx} + 2f_y f_{xy})(1+f_y^2)}{(1+f_x^2+f_y^2)^2}$

$= \frac{-f_x(1+f_y^2+f_x^2 f_y^2) f_{xx} + f_x^2 f_y (1+f_x^2-f_y^2) f_{xy} + f_x^3 f_y^2 f_{yy}}{(1+f_x^2+f_y^2)^3}$

6. 设 (u, v) 是正交曲线参数, 则 $F = M = 0$.

Codazzi 方程:
$$\begin{cases} \left(\frac{L}{\sqrt{E}}\right)_v = N \frac{(\sqrt{E})_v}{G} & \textcircled{1} \\ \left(\frac{N}{\sqrt{G}}\right)_u = L \frac{(\sqrt{G})_u}{E} & \textcircled{2} \end{cases}$$

$\textcircled{1}$: 左边 $= \frac{L_v}{\sqrt{E}} - \frac{LE_v}{2E^{\frac{3}{2}}}$, $\textcircled{2}$: 右边 $= \frac{N}{G} \cdot \frac{E_v}{2\sqrt{E}}$

同理,
 $\textcircled{2} \Rightarrow N_u = HG_u$.

7. 设曲面 S 无脐点, 则 $k_1 \neq k_2$. 不妨设 $k_1 > H > k_2$.

取 (u, v) 是 S 的正交曲线参数, 则
$$\begin{cases} L_v = HE_v, \\ N_u = HG_u. \end{cases}$$

故 $L = HE + f(u)$, $N = HG + g(v)$.

则 $k_1 = \frac{L}{E} = H + \frac{f(u)}{E}$, $k_2 = \frac{N}{G} = H + \frac{g(v)}{G}$.

则 $f(u) < 0$ 且 $g(v) < 0$. $\frac{f(u)}{E} = -\frac{g(v)}{G} =: \lambda$

$E = \lambda f(u)$, $G = -\lambda g(v)$.

$$I = \lambda(f(u) du^2 - g(v) dv^2) = \lambda(d\tilde{u}^2 + d\tilde{v}^2) \quad \begin{cases} \tilde{u} = \int \sqrt{f(u)} du \\ \tilde{v} = \int \sqrt{-g(v)} dv \end{cases}$$

$$\begin{aligned} II &= L du^2 + N dv^2 = (\lambda H + 1) f(u) du^2 + (-\lambda H + 1) g(v) dv^2 \\ &= (\lambda H + 1) d\tilde{u}^2 + (\lambda H - 1) d\tilde{v}^2 \end{aligned}$$

注 S 可能有脐点, 但非全脐点曲面.

8. 由 E, F, G, L, M, N 均为常数, k_1, k_2 均为常数

由例 6.4, S 为平面或球面或圆柱面.

但球面: 第一类基本量不为常数 故 S 为平面或圆柱面.

9. Gauss 方程
$$\frac{LN-M^2}{EG} = -\frac{1}{\sqrt{EG}} \left(\frac{\sqrt{E}v}{\sqrt{G}} + \frac{\sqrt{G}u}{\sqrt{E}} \right)$$

Codazzi 方程
$$\begin{cases} L_v = HE_v, \\ N_u = HG_u. \end{cases}$$

1) Gauss 方程 $-1 \neq 0$ 不满足.

2) Gauss 方程
$$\frac{\cos^2 u}{\cos^2 u} = 1 = -\frac{1}{\sqrt{\cos^2 u}} \left(\frac{\sqrt{\cos^2 u}}{\cos^2 u} \right) \quad \text{不满足}$$

Codazzi 方程
$$\begin{cases} 0 = L_v = 0 = E_v \quad \checkmark \\ N_u = 0 \neq -\frac{\cos^4 u + 1}{\cos^2 u} \sin 2u = HG_u \quad \times \end{cases}$$

$$\begin{cases} H = \frac{LG - 2MF + NE}{EG - F^2} = \frac{\cos^4 u + 1}{\cos^2 u} \\ G_u = -\sin 2u \end{cases}$$

$$10. I = \left(1 + \frac{F_x^2}{F_z^2}\right) dx^2 + \frac{2F_x F_y}{F_z^2} dx dy + \left(1 + \frac{F_y^2}{F_z^2}\right) dy^2, \quad F_z \neq 0.$$

$$II = \frac{\text{sgn}(F_z)}{F_z \sqrt{F_x^2 + F_y^2 + F_z^2}} \left((-F_z^3 F_{xx} + 2F_x F_z F_{xz} - F_x^2 F_{zz}) dx^2 + 2(-F_z^2 F_{xy} + F_y F_z F_{xz} + F_x F_z F_{yz} - F_x F_y F_{zz}) dx dy \right. \\ \left. + (-F_z^2 F_{yy} + 2F_y F_z F_{yz} - F_y^2 F_{zz}) dy^2 \right)$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{1}{F_z^2 (F_x^2 + F_y^2 + F_z^2)^2} \left(F_x^2 F_{zz}^2 + F_y^2 F_{zz}^2 F_x F_{zz} + F_x^4 F_{yy} F_{zz} - 2F_y F_z^3 F_{xx} F_{yz} \right. \\ \left. - 2F_x^3 F_z F_{xz} F_{yy} + 2F_x F_y F_z^2 F_{xz} F_{yz} - F_z^4 F_{xy}^2 - F_y^2 F_z^2 F_{xz}^2 - F_x^2 F_z^2 F_{yz}^2 \right. \\ \left. + 2F_y F_z^3 F_{xy} F_{xz} + 2F_x F_z^3 F_{xy} F_{yz} - 2F_x F_y F_z^2 F_{xy} F_{zz} \right)$$

$$11. \text{ 设 } S: r = r(u, v). \text{ 令 } e_1 = \frac{r_u}{\sqrt{E}}, e_2 = \frac{r_v}{\sqrt{G}}, e_3 = n = e_1 \wedge e_2.$$

$$\text{由正交标架的运动方程} \begin{cases} dr = w_1 e_1 + w_2 e_2 \\ de_1 = w_{12} e_2 + w_{13} e_3 \\ de_2 = w_{21} e_1 + w_{23} e_3 \\ de_3 = w_{31} e_1 + w_{32} e_2 \end{cases}$$

$$w_1 = \sqrt{E} du, w_2 = \sqrt{G} dv. \begin{cases} w_{12} = -\frac{\sqrt{E} r_v}{\sqrt{G}} du + \frac{\sqrt{G} r_u}{\sqrt{E}} dv \\ w_{13} = \frac{L}{\sqrt{E}} du + \frac{M}{\sqrt{E}} dv \\ w_{23} = \frac{M}{\sqrt{G}} du + \frac{N}{\sqrt{G}} dv \end{cases}$$

$$\text{由} \begin{cases} e_{1u} = -\frac{E_v}{2\sqrt{EG}} e_2 + \frac{L}{\sqrt{E}} e_3 = \frac{1}{1+u^2} e_3 \quad (1) \\ e_{2u} = \frac{E_v}{2\sqrt{EG}} e_1 + \frac{M}{\sqrt{G}} e_3 = 0 \quad (2) \\ e_{3u} = -\frac{L}{\sqrt{E}} e_1 - \frac{M}{\sqrt{G}} e_2 = -\frac{1}{1+u^2} e_1 \quad (3) \\ e_{1v} = \frac{G_u}{2\sqrt{EG}} e_2 + \frac{M}{\sqrt{E}} e_3 = \frac{1}{\sqrt{1+u^2}} e_2 \quad (4) \\ e_{2v} = -\frac{G_u}{2\sqrt{EG}} e_1 + \frac{N}{\sqrt{G}} e_3 = -\frac{1}{\sqrt{1+u^2}} e_1 + \frac{u}{\sqrt{1+u^2}} e_3 \quad (5) \\ e_{3v} = -\frac{M}{\sqrt{E}} e_1 - \frac{N}{\sqrt{G}} e_2 = -\frac{u}{\sqrt{1+u^2}} e_2 \quad (6) \end{cases}$$

$$(1) \quad e_{1u} = -\frac{E_v}{2\sqrt{EG}} e_2 + \frac{L}{\sqrt{E}} e_3 = \frac{1}{1+u^2} e_3$$

$$(2) \quad e_{2u} = \frac{E_v}{2\sqrt{EG}} e_1 + \frac{M}{\sqrt{G}} e_3 = 0$$

$$(3) \quad e_{3u} = -\frac{L}{\sqrt{E}} e_1 - \frac{M}{\sqrt{G}} e_2 = -\frac{1}{1+u^2} e_1$$

$$(4) \quad e_{1v} = \frac{G_u}{2\sqrt{EG}} e_2 + \frac{M}{\sqrt{E}} e_3 = \frac{1}{\sqrt{1+u^2}} e_2$$

$$(5) \quad e_{2v} = -\frac{G_u}{2\sqrt{EG}} e_1 + \frac{N}{\sqrt{G}} e_3 = -\frac{1}{\sqrt{1+u^2}} e_1 + \frac{u}{\sqrt{1+u^2}} e_3$$

$$(6) \quad e_{3v} = -\frac{M}{\sqrt{E}} e_1 - \frac{N}{\sqrt{G}} e_2 = -\frac{u}{\sqrt{1+u^2}} e_2$$

由②, $e_2 = e_2(v)$. 设弧长参数曲线 $C: \tilde{r} = \tilde{r}(v)$ 且 $\dot{\tilde{r}} = t = e_2(v)$

其 Frenet 标架 $\tilde{r}, t, n, b$. 则
$$\begin{cases} e_1 = n \cos \theta + b \sin \theta & (1) \\ e_2 = t & (8) \\ e_3 = +n \sin \theta - b \cos \theta & (9) \end{cases}$$

由⑦,
$$e_{1v} = -\kappa \cos \theta e_2 + (\tau + \theta_v) e_3$$

$$\stackrel{(4)}{=} \frac{1}{\sqrt{1+u^2}} e_2$$

$$\Rightarrow \begin{cases} \tau + \theta_v = 0 & (10) \\ -\kappa \cos \theta = \frac{1}{\sqrt{1+u^2}} & (11) \end{cases}$$

由⑧,
$$e_{2v} = \kappa n = \kappa \cos \theta e_1 + \kappa \sin \theta e_2$$

$$\stackrel{(5)}{=} -\frac{1}{\sqrt{1+u^2}} e_1 + \frac{u}{\sqrt{1+u^2}} e_2$$

$$\Rightarrow \begin{cases} \kappa \cos \theta = -\frac{1}{\sqrt{1+u^2}} \\ +\kappa \sin \theta = \frac{u}{\sqrt{1+u^2}} \end{cases} \Rightarrow \begin{cases} \cos \theta = -\frac{1}{\kappa \sqrt{1+u^2}} \\ \sin \theta = \frac{u}{\kappa \sqrt{1+u^2}} \end{cases}$$

$$\Rightarrow \kappa = \frac{1}{\sqrt{1+u^2}} \Rightarrow \begin{cases} \cos \theta = -\frac{1}{\sqrt{1+u^2}} \\ \sin \theta = \frac{u}{\sqrt{1+u^2}} \end{cases} \Rightarrow \theta_v = 0.$$

(10) $\tau = 0 \Rightarrow C$ 是单位圆 可设 $C: \tilde{r} = (\cos v, \sin v, 0)$

则 $t = (-\sin v, \cos v, 0)$, $n = (-\cos v, -\sin v, 0)$, $b = (0, 0, 1)$

$$\Rightarrow \begin{cases} e_1 = (-\cos \theta \cos v, -\cos \theta \sin v, \sin \theta) = \frac{1}{\sqrt{1+u^2}} (\cos v, \sin v, +u) \\ e_2 = (-\sin v, \cos v, 0) \end{cases}$$

$$\Rightarrow \begin{cases} r_u = \sqrt{E} e_1 = (\cos v, \sin v, u) \\ r_v = \sqrt{G} e_2 = (-u \sin v, u \cos v, 0) \end{cases} \Rightarrow r = (u \cos v, u \sin v, \frac{u^2}{2})$$

4-8 故 S^2 抛物线 $\begin{cases} z = \frac{1}{2}x^2 \\ y = 0 \end{cases}$ 绕 z 轴旋转得到: 抛物面.

12. 1) 设 ρ, ψ 是曲面 S 的第一、二基本形式. 令 ψ, ψ 是 Gauss 曲率

$\psi = 1/\lambda = 0$, (u, v) 是正则曲率参数. 令 E, G, λ 满足

$$\begin{cases} \lambda^2 = -\frac{1}{\sqrt{EG}} \left(\left(\frac{\sqrt{E}}{\sqrt{G}} \right)_v + \left(\frac{\sqrt{G}}{\sqrt{E}} \right)_u \right) \\ \frac{\partial \lambda E}{\partial v} = \lambda \frac{\partial E}{\partial v} \Rightarrow E \frac{\partial \lambda}{\partial v} = 0 \\ \frac{\partial \lambda G}{\partial u} = \lambda \frac{\partial G}{\partial u} \Rightarrow G \frac{\partial \lambda}{\partial u} = 0 \end{cases} \quad \begin{cases} H = \lambda \\ K = \lambda^2 \end{cases}$$

S 全脐点.

$$\Rightarrow \begin{cases} \frac{\partial \lambda}{\partial v} = 0 \\ \frac{\partial \lambda}{\partial u} = 0 \end{cases} \Rightarrow \lambda = \text{常数, 且 } \lambda^2 = -\frac{1}{\sqrt{EG}} \left(\left(\frac{\sqrt{E}}{\sqrt{G}} \right)_v + \left(\frac{\sqrt{G}}{\sqrt{E}} \right)_u \right)$$

$$\text{若 } E = G, \text{ 则 } \lambda^2 = -\frac{1}{\sqrt{E^2}} \left((\ln \sqrt{E})_{uu} + (\ln \sqrt{E})_{vv} \right)$$

$$= -\frac{1}{E} \Delta \ln \sqrt{E} \Leftrightarrow \Delta \ln \sqrt{E} = -\lambda^2 E$$

13. $r(u, v) = (u \cos v, u \sin v, f(u)) \quad (u > 0)$

$$\begin{cases} r_u = (\cos v, \sin v, f'(u)) \\ r_v = (-u \sin v, u \cos v, 0) \end{cases} \quad \begin{cases} E = 1 + f'(u)^2 \\ F = 0 \\ G = u^2 \end{cases} \quad I = (1 + f'(u)^2) du^2 + u^2 dv^2$$

$$r_u \wedge r_v = (-u f'(u) \cos v, -u f'(u) \sin v, u) \quad n = \frac{1}{\sqrt{1 + f'(u)^2}} (-f'(u) \cos v, -f'(u) \sin v, 1)$$

$$\begin{cases} r_{uu} = (0, 0, f''(u)) \\ r_{uv} = (-\sin v, \cos v, 0) \\ r_{vv} = (-u \cos v, -u \sin v, 0) \end{cases} \quad \begin{cases} L = \frac{f''(u)}{\sqrt{1 + f'(u)^2}} \\ M = 0 \\ N = \frac{u f'(u)}{\sqrt{1 + f'(u)^2}} \end{cases}$$

$$\hat{e}_1 = \frac{r_u}{\sqrt{1 + f'(u)^2}}, \quad \hat{e}_2 = \frac{r_v}{u}, \quad \hat{e}_3 = \frac{r_u \wedge r_v}{u} \quad \begin{cases} w_1 = \sqrt{1 + f'(u)^2} du \\ w_2 = u dv \end{cases}$$

$$w_{12} = -\frac{(\sqrt{E})_v}{\sqrt{G}} du + \frac{(\sqrt{G})_u}{\sqrt{E}} dv = \frac{1}{\sqrt{H f'(u)^2}} dv$$

$$w_{13} = \frac{L}{\sqrt{E}} du + \frac{M}{\sqrt{E}} dv = \frac{f'(u)}{1+f'(u)^2} du$$

$$w_{23} = \frac{M}{\sqrt{G}} du + \frac{N}{\sqrt{G}} dv = \frac{f'(u)}{\sqrt{H f'(u)^2}} dv$$

14. $\bar{w}_{12} = w_{12} + d\theta \Rightarrow d\bar{w}_{12} = dw_{12} \quad w_1 \wedge w_2 = \bar{w}_1 \wedge \bar{w}_2$

to $\frac{d\bar{w}_{12}}{\bar{w}_1 \wedge \bar{w}_2} = \frac{dw_{12}}{w_1 \wedge w_2}$

15. $r(u, v) = (a \cos u \cos v, a \cos u \sin v, a \sin u) \quad (-\frac{\pi}{2} < u < \frac{\pi}{2}, 0 < v < 2\pi)$

$$\begin{cases} r_u = (-a \sin u \cos v, -a \sin u \sin v, a \cos u) \\ r_v = (-a \cos u \sin v, a \cos u \cos v, 0) \end{cases} \begin{cases} E = a^2 \\ F = 0 \\ G = a^2 \cos^2 u \end{cases}$$

$$r_u \wedge r_v = (-a^2 \cos^2 u \cos v, -a^2 \cos^2 u \sin v, -a^2 \cos u \sin u)$$

$$n = (-\cos u \cos v, -\cos u \sin v, -\sin u)$$

$$\begin{cases} r_{uu} = (-a \cos u \cos v, -a \cos u \sin v, -a \sin u) \\ r_{uv} = (a \sin u \sin v, -a \sin u \cos v, 0) \\ r_{vv} = (-a \cos u \cos v, -a \cos u \sin v, 0) \end{cases} \begin{cases} L = a \\ M = 0 \\ N = a^2 \cos^2 u \end{cases}$$

1) $\begin{cases} e_1 = \frac{r_u}{a} \\ e_2 = \frac{r_v}{a \cos u} \\ e_3 = n \end{cases}$

(2) $w_1 = a du, w_2 = a \cos u dv$

$$\begin{cases} w_{12} = -\sin u dv \\ w_{13} = du \\ w_{23} = \cos u dv \end{cases}$$

4-10 3) $\Pi = w_1 w_{13} + w_2 w_{23} = a du^2 + a \cos^2 u dv^2$

16. 设 e_1, e_2 是 S 上正交向量场. $\tilde{n} = \pm n$.

(a) $\tilde{n} = n$, 取 $e_3 = n$. 设 $dr = w_1 e_1 + w_2 e_2$, $\omega(e_i) = k_i e_i$.

$$\lambda - dn = \omega(dr) = w_1 \omega(e_1) + w_2 \omega(e_2) = k_1 w_1 e_1 + k_2 w_2 e_2.$$

$$dn = k_1 w_1 e_1 + k_2 w_2 e_2.$$

$$\tilde{w}_1 e_1 + \tilde{w}_2 e_2 = d\tilde{r} = (1 - \lambda k_1) w_1 e_1 + (1 - \lambda k_2) w_2 e_2$$

$$\begin{cases} \tilde{w}_1 = (1 - \lambda k_1) w_1 \\ \tilde{w}_2 = (1 - \lambda k_2) w_2 \end{cases}$$

$$\tilde{w}_1 \wedge \tilde{w}_2 = (1 - \lambda k_1)(1 - \lambda k_2) w_1 \wedge w_2 = (1 - 2\lambda H + \lambda^2 K) w_1 \wedge w_2$$

$$\tilde{w}_{12} = \langle de_1, e_2 \rangle = w_{12}, \quad K = - \frac{dw_{12}}{w_1 \wedge w_2}$$

$$\tilde{K} = - \frac{d\tilde{w}_{12}}{\tilde{w}_1 \wedge \tilde{w}_2} = - \frac{dw_{12}}{(1 - 2\lambda H + \lambda^2 K) w_1 \wedge w_2} = \frac{K}{1 - 2\lambda H + \lambda^2 K}.$$

$$H = + \frac{w_1 \wedge w_{23} - w_2 \wedge w_{13}}{2 w_1 \wedge w_2}$$

$$\tilde{H} = + \frac{\tilde{w}_1 \wedge \tilde{w}_{23} - \tilde{w}_2 \wedge \tilde{w}_{13}}{2 \tilde{w}_1 \wedge \tilde{w}_2} = + \frac{(1 - \lambda k_1) w_1 \wedge w_{23} - (1 - \lambda k_2) w_2 \wedge w_{13}}{2 (1 - 2\lambda H + \lambda^2 K) w_1 \wedge w_2}$$

$$= + \frac{w_1 \wedge w_{23} - w_2 \wedge w_{13}}{2 (1 - 2\lambda H + \lambda^2 K) w_1 \wedge w_2} - \frac{\lambda w_{13} \wedge w_{23} - \lambda w_{23} \wedge w_{13}}{2 (1 - 2\lambda H + \lambda^2 K) w_1 \wedge w_2}$$

$$= \frac{H}{1 - 2\lambda H + \lambda^2 K} - \frac{\lambda K}{1 - 2\lambda H + \lambda^2 K}$$

$$= \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}.$$

17. 由例 6.5 得非脐点邻域. 若 P 是脐点, 若 P 邻域, 全脐点

则 S 是平面, 而 P 是脐点邻域时, 脐点不决定 S 的形状

故 S 是直纹面且 Gauss 曲率为 0.

18. $K=0$.

19. $\left\{ \begin{array}{l} dw_{13} = w_{12} \wedge w_{23} \\ dw_{23} = w_{21} \wedge w_{13} \end{array} \right. \quad \left\{ \begin{array}{l} w_{12} = -\frac{\sqrt{E}v}{\sqrt{G}} du + \frac{\sqrt{G}u}{\sqrt{E}} dv \\ w_{13} = \frac{L}{E} du + \frac{M}{\sqrt{E}} dv \\ w_{23} = \frac{N}{\sqrt{G}} du + \frac{X}{\sqrt{G}} dv \end{array} \right.$

Codazzi $\tilde{\sigma} \tilde{\sigma} \tilde{\sigma}$

$\Rightarrow \left\{ \begin{array}{l} \left(\frac{L}{\sqrt{E}}\right)_v - \left(\frac{M}{\sqrt{E}}\right)_u - N \frac{\sqrt{E}v}{G} - M \frac{\sqrt{G}u}{\sqrt{E}G} = 0, \\ \left(\frac{N}{\sqrt{G}}\right)_u - \left(\frac{M}{\sqrt{G}}\right)_v - L \frac{\sqrt{G}u}{E} - M \frac{\sqrt{E}v}{\sqrt{E}G} = 0. \end{array} \right.$

20. $\left\{ \begin{array}{l} w_{13} = k_1 w_1 \\ w_{23} = k_2 w_2 \end{array} \right. \quad \left\{ \begin{array}{l} dw_{13} = w_{12} \wedge w_{23} \\ dw_{23} = w_{21} \wedge w_{13} \end{array} \right.$

Codazzi $\tilde{\sigma} \tilde{\sigma} \tilde{\sigma}$

$\left\{ \begin{array}{l} dk_1 \wedge w_1 + k_1 dw_1 = w_{12} \wedge k_2 w_2 \\ dk_2 \wedge w_2 + k_2 dw_2 = w_{21} \wedge k_1 w_1 \end{array} \right.$

$\updownarrow$

$\forall \left\{ \begin{array}{l} dw_1 = w_2 \wedge w_{21} = w_{12} \wedge w_2 \\ dw_2 = w_1 \wedge w_{12} = w_{21} \wedge w_1 \end{array} \right. \left\{ \begin{array}{l} dk_1 \wedge w_1 = (k_2 - k_1) w_{12} \wedge w_2, \\ dk_2 \wedge w_2 = (k_1 - k_2) w_{21} \wedge w_1. \end{array} \right.$

习题五

1. (1) $K = -\frac{1}{\sqrt{EG}} \left(\left(\frac{\sqrt{E}}{\sqrt{G}} \right)_v + \left(\frac{\sqrt{G}}{\sqrt{E}} \right)_u \right) = 0$ (2) $K = 1$

(3) $K = 4c$ (4) $K = 0 = -\frac{dw_{12}}{w_1 \wedge w_2}$

2. 设 $\tilde{I} = \tilde{w}_1^2 + \tilde{w}_2^2$. $\hat{w}_i = \sqrt{\lambda} \tilde{w}_i$, 则 $I = \lambda \tilde{I} = w_1^2 + w_2^2$.

$$\begin{cases} d\tilde{w}_1 = \tilde{w}_2 \wedge \tilde{w}_2 \\ d\tilde{w}_2 = \tilde{w}_1 \wedge \tilde{w}_1 \end{cases} \Leftrightarrow \begin{cases} dw_1 = \tilde{w}_2 \wedge w_2 \\ dw_2 = \tilde{w}_1 \wedge w_1 \end{cases} \Rightarrow w_{12} = \tilde{w}_{12}$$

$$K = \frac{dw_{12}}{w_1 \wedge w_2} = -\frac{1}{\lambda} \frac{d\tilde{w}_{12}}{\tilde{w}_1 \wedge \tilde{w}_2} = \frac{1}{\lambda} \tilde{K}$$

3. (1) $D = E^2$. $\begin{cases} r_u = (a, 0, au) \\ r_v = (0, b, bv) \end{cases} \Rightarrow \begin{cases} E = a^2(1+u^2) \\ F = abuv \\ G = b^2(1+v^2) \end{cases} \Rightarrow ab \neq 0$

$$\begin{aligned} r_u \wedge r_v &= (abu, -abv, ab) \quad n = \frac{\text{sgn}(ab)}{\sqrt{1+u^2+v^2}} (-u, -v, 1) \\ \begin{cases} r_{uu} &= (0, 0, a) \\ r_{uv} &= 0 \\ r_{vv} &= (0, 0, b) \end{cases} \quad \begin{cases} L = \frac{\text{sgn}(ab)a}{\sqrt{1+u^2+v^2}} \\ M = 0 \\ N = \frac{\text{sgn}(ab)b}{\sqrt{1+u^2+v^2}} \end{cases} \quad K = \frac{LN-M^2}{EGF^2} = \frac{1}{ab(1+u^2+v^2)^2} \\ &= \frac{1}{a^2b^2(1+u^2+v^2)^2} = \tilde{K}(u, v) \end{aligned}$$

(2) 假设 S 与 $\tilde{S}$ 之间有等距变换: $D \xrightarrow{\sigma} \tilde{D} \quad (u, v) \mapsto (\tilde{u}(u, v), \tilde{v}(u, v))$ 则

$$\frac{1}{ab(1+u^2+v^2)} = K(u, v) = \tilde{K}(\tilde{u}, \tilde{v}) = \frac{1}{a\tilde{b}(1+\tilde{u}^2+\tilde{v}^2)}$$

$$\Leftrightarrow \frac{1+\tilde{u}^2+\tilde{v}^2}{1+u^2+v^2} = c := \frac{ab}{a\tilde{b}} \quad \Leftrightarrow \text{一阶偏导数均为0且在一点值为} c$$

$$\Leftrightarrow \begin{cases} \frac{2(\tilde{u} \frac{\partial \tilde{u}}{\partial u} + \tilde{v} \frac{\partial \tilde{v}}{\partial u})(1+\tilde{u}^2+\tilde{v}^2) - 2u(1+\tilde{u}^2+\tilde{v}^2)}{(1+u^2+v^2)^2} = 0 \\ \frac{2(\tilde{u} \frac{\partial \tilde{u}}{\partial v} + \tilde{v} \frac{\partial \tilde{v}}{\partial v})(1+\tilde{u}^2+\tilde{v}^2) - 2v(1+\tilde{u}^2+\tilde{v}^2)}{(1+u^2+v^2)^2} = 0 \end{cases} \quad \& \quad 1+\tilde{u}^2(0,0)+\tilde{v}^2(0,0)=c$$

$$\Leftrightarrow (*) \begin{cases} \tilde{u} \frac{\partial \tilde{u}}{\partial u} + \tilde{v} \frac{\partial \tilde{v}}{\partial u} = cu \\ \tilde{v} \frac{\partial \tilde{u}}{\partial v} + \tilde{v} \frac{\partial \tilde{v}}{\partial v} = cv \end{cases} \quad \& \quad 1+\tilde{u}^2(0,0)+\tilde{v}^2(0,0)=c$$

$$O=(0,0)$$

$$\text{取 } (u,v)=(0,0) \Rightarrow \begin{cases} \tilde{u}(0,0)=0 \\ \tilde{v}(0,0)=0 \end{cases} \text{ 且 } C=1 \Leftrightarrow \sigma(0)=0 \text{ 且 } ab=\tilde{a}\tilde{b}$$

将(*)再求一次偏导数

$$\begin{cases} (\frac{\partial \tilde{u}}{\partial u})^2 + \tilde{u} \frac{\partial^2 \tilde{u}}{\partial u^2} + (\frac{\partial \tilde{v}}{\partial u})^2 + \tilde{v} \frac{\partial^2 \tilde{v}}{\partial u^2} = 1 \\ \frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{u}}{\partial v} + \tilde{u} \frac{\partial^2 \tilde{u}}{\partial u \partial v} + \frac{\partial \tilde{v}}{\partial u} \frac{\partial \tilde{v}}{\partial v} + \tilde{v} \frac{\partial^2 \tilde{v}}{\partial u \partial v} = 0 \\ (\frac{\partial \tilde{u}}{\partial v})^2 + \tilde{u} \frac{\partial^2 \tilde{u}}{\partial v^2} + (\frac{\partial \tilde{v}}{\partial v})^2 + \tilde{v} \frac{\partial^2 \tilde{v}}{\partial v^2} = 1 \end{cases}$$

$$\text{取 } (u,v)=(0,0)$$

$$\begin{cases} (\frac{\partial \tilde{u}}{\partial u}(0))^2 + (\frac{\partial \tilde{v}}{\partial v}(0))^2 = 1 \\ \frac{\partial \tilde{u}}{\partial u}(0) \frac{\partial \tilde{u}}{\partial v}(0) + \frac{\partial \tilde{v}}{\partial u}(0) \frac{\partial \tilde{v}}{\partial v}(0) = 0 \\ (\frac{\partial \tilde{u}}{\partial v}(0))^2 + (\frac{\partial \tilde{v}}{\partial v}(0))^2 = 1 \end{cases} \Leftrightarrow J_0 \text{ 是正交阵}$$

$$\Leftrightarrow J_0 = \begin{pmatrix} \cos \theta & \sin \theta \\ -\delta \sin \theta & \delta \cos \theta \end{pmatrix}, \theta \text{ 是固定角度}, \delta = \pm 1$$

由(*)是等距,

$$\begin{pmatrix} a^2(1+u^2) & abuv \\ abuv & b^2(1+v^2) \end{pmatrix} = J_0 \begin{pmatrix} \tilde{a}^2(1+\tilde{u}^2) & \tilde{a}\tilde{b}\tilde{u}\tilde{v} \\ \tilde{a}\tilde{b}\tilde{u}\tilde{v} & \tilde{b}^2(1+\tilde{v}^2) \end{pmatrix} J_0^t$$

$$\text{取 } (u,v)=(0,0) \Rightarrow \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\delta \sin \theta & \delta \cos \theta \end{pmatrix} \begin{pmatrix} \tilde{a}^2 & 0 \\ 0 & \tilde{b}^2 \end{pmatrix} \begin{pmatrix} \cos \theta & -\delta \sin \theta \\ \sin \theta & \delta \cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{a}^2 \cos^2 \theta + \tilde{b}^2 \sin^2 \theta & \delta(\tilde{b}^2 - \tilde{a}^2) \sin \theta \cos \theta \\ \delta(\tilde{b}^2 - \tilde{a}^2) \sin \theta \cos \theta & \tilde{a}^2 \sin^2 \theta + \tilde{b}^2 \cos^2 \theta \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \tilde{a}^2 \cos^2 \theta + \tilde{b}^2 \sin^2 \theta = a^2 \\ (\tilde{b}^2 - \tilde{a}^2) \sin \theta \cos \theta = 0 \\ \tilde{a}^2 \sin^2 \theta + \tilde{b}^2 \cos^2 \theta = b^2 \end{cases} \Leftrightarrow \begin{cases} ab = \tilde{a}\tilde{b} \\ (\tilde{a}^2, \tilde{b}^2) = (a^2, b^2) \text{ 或 } (b^2, a^2) \end{cases}$$

在此条件下, S 与 $\tilde{S}$ 之间都存在等距:

$$D \rightarrow \tilde{D}$$

$$(u,v) \mapsto (\tilde{u}, \tilde{v}) = (\pm u, \pm v) \text{ 或 } (\pm v, \pm u)$$

7. (1). $w_1 = \sqrt{u+v} du$, $w_2 = \sqrt{u+v} dv \Rightarrow w_1 \wedge w_2 = (u+v) du dv$.

$$\begin{cases} dw_1 = -\frac{v'(u)}{2\sqrt{u+v}} du dv = -\frac{v'(u)}{2(u+v)^{\frac{3}{2}}} w_1 \wedge w_2 \\ dw_2 = \frac{u'(v)}{2\sqrt{u+v}} du dv = \frac{u'(v)}{2(u+v)^{\frac{3}{2}}} w_1 \wedge w_2 \end{cases}$$

$$\Rightarrow w_2 = -\frac{v'(u)}{2(u+v)^{\frac{3}{2}}} w_1 + \frac{u'(v)}{2(u+v)^{\frac{3}{2}}} w_2 = -\frac{v'(u) du}{2(u+v)} + \frac{u'(v) dv}{2(u+v)}$$

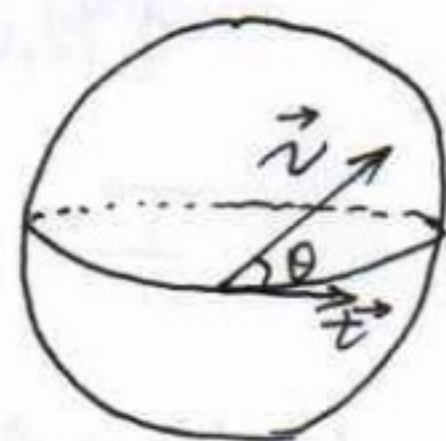
$$K = -\frac{dw_{12}}{w_1 \wedge w_2} = -\frac{(u''+v'')(u+v) - u'^2 - v'^2}{2(u+v)^3}$$

(2) $w_1 = \frac{du - 2v dv}{2\sqrt{u-v^2}}$, $w_2 = dv$ $w_1 \wedge w_2 = \frac{1}{2\sqrt{u-v^2}} du dv$

$$\begin{cases} dw_1 = \frac{2v}{(u-v^2)^{\frac{3}{2}}} du dv = \frac{2v}{u-v^2} w_1 \wedge w_2 \\ dw_2 = 0 \end{cases} \Rightarrow w_{12} = \frac{2v w_1}{u-v^2} = \frac{v du - 2v^2 dv}{(u-v^2)^{\frac{3}{2}}} \\ \Rightarrow dw_{12} = -\frac{du dv}{(u-v^2)^{\frac{3}{2}}}$$

$$K = -\frac{dw_{12}}{w_1 \wedge w_2} = \frac{2}{u-v^2}$$

8. 赤道是测地线, 其切向单位向量 $\vec{v}$ 是平行的.



对任意切向量 v , 它平移(沿赤道)保持与 $\vec{v}$ 的内积, 从而夹角不变.

由平移的惟一性, v 平移得到: 切向量场 $\vec{v}$ 与 $\vec{v}$ 保持固定夹角 $0 \in [0, \pi]$.

9. $\vec{v}$ 平行 $\Leftrightarrow 0 = \frac{Dv}{dt} = \left(\frac{df^\alpha}{dt} + T_{\beta\gamma}^\alpha f^\beta \frac{du^\gamma}{dt} \right) \pi_\alpha$

$$\Leftrightarrow \begin{cases} -\frac{1}{2E^{\frac{3}{2}}} \frac{du^\alpha}{dt} + T_{1\alpha}^1 \frac{1}{\sqrt{E}} \frac{du^\alpha}{dt} = 0 & (\text{恒成立}) \\ T_{1\alpha}^2 \frac{du^\alpha}{dt} = 0 \end{cases}$$

$$\Leftrightarrow T_{1\alpha}^2 \frac{du^\alpha}{dt} = 0$$

$$10. 1) I = a^2 du^2 + a^2 \cos^2 u dv^2$$

$$\begin{aligned} \text{Liouville's} \Rightarrow k_g &= \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \ln E}{\partial v} \cos \theta + \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin \theta \\ &= \frac{d\theta}{ds} - \frac{a \sin u}{a \cos u} \sin \theta \quad \left(\sin \theta = \sqrt{G} \frac{dv}{ds} = a \cos u \frac{dv}{ds} \right) \\ &= \frac{d\theta}{ds} - \sin u \frac{dv}{ds} \end{aligned}$$

$$\begin{aligned} (2) \text{ 纬圆} = v\text{-线}, \text{ 由 (1)}, k_g &= \frac{d\theta}{ds} - \sin u_0 \frac{dv}{ds} \quad \left(\frac{dv}{ds} = \frac{1}{a \cos u_0} \right) \\ C_{u_0}: r(u, v) & \quad \theta = \frac{\pi}{2} \\ &= -\sin u_0 \frac{1}{a \cos u_0} \\ &= -\frac{1}{a} \tan u_0. \end{aligned}$$

$$11. \quad r(u, v) = (f(u) \cos v, f(u) \sin v, g(u)) \quad (f > 0)$$

$$I = (f'^2 + g'^2) du^2 + f^2 dv^2$$

$$\text{纬线} = v\text{-线} \quad k_g(C_{u_0}) = \frac{1}{2\sqrt{E(u_0)}} \frac{\partial \ln G(u_0, v)}{\partial u} = \frac{f'}{f\sqrt{f'^2 + g'^2}}$$

$$\text{经线} = u\text{-线} \quad k_g(C_{v_0}) = -\frac{1}{2\sqrt{G(u, v_0)}} \frac{\partial \ln E(u, v_0)}{\partial v} = 0$$

$$12. 1) \lambda_1 = \left\langle \frac{de_1}{ds}, e_2 \right\rangle = k_g, \quad \lambda_2 = \left\langle \frac{de_1}{ds}, e_3 \right\rangle = k_n$$

$$\begin{aligned} (2) \lambda_3 &= \left\langle \frac{de_2}{ds}, e_3 \right\rangle = \tau_g, \quad e_2 = e_3 \wedge e_1, \quad \dot{e}_2 = \dot{e}_3 \wedge e_1 + e_3 \wedge \dot{e}_1 \\ &= \langle \dot{e}_3 \wedge e_1, e_3 \rangle + \langle e_3 \wedge \dot{e}_1, e_3 \rangle \\ &= \langle \dot{e}_3, e_1, e_3 \rangle + \langle e_3, e_1, \dot{e}_3 \rangle \\ &= \langle e_3, \dot{e}_3, e_1 \rangle = (n, \dot{n}, \hat{r}) \end{aligned}$$

13. 可设 t 是 $C: r(t)$ 的弧长参数. 令 $e_1 = v$, $e_2 = n \wedge v$, $e_3 = n$.

则 $w = \cos\theta e_1 + \sin\theta e_2$, $n \wedge w = -\sin\theta e_1 + \cos\theta e_2$.

$$\frac{Dw}{dt} = \frac{d\theta}{dt} (-\sin\theta e_1 + \cos\theta e_2) + \cos\theta \frac{De_1}{dt} + \sin\theta \frac{De_2}{dt}$$

$$\begin{aligned} \langle n \wedge w, \cdot \rangle, \quad \left\langle \frac{Dw}{dt}, n \wedge w \right\rangle &= \frac{d\theta}{dt} + \cos^2\theta \left\langle \frac{De_1}{dt}, e_2 \right\rangle - \sin^2\theta \left\langle \frac{De_2}{dt}, e_1 \right\rangle \\ &\stackrel{\langle \frac{De_1}{dt}, e_2 \rangle + \langle \frac{De_2}{dt}, e_1 \rangle = \frac{d\theta}{dt}}{=} \frac{d\theta}{dt} + \left\langle \frac{Dv}{dt}, n \wedge w \right\rangle \end{aligned}$$

14. $I = (f'^2 + g'^2) du^2 + f^2 dv^2$

则曲线 $C \subset \frac{H^2}{f^2}$: $\begin{cases} \frac{d\theta}{ds} = -\frac{f'}{f\sqrt{f'^2+g'^2}} \sin\theta \\ \frac{du}{ds} = \frac{\cos\theta}{\sqrt{f'^2+g'^2}} \\ f \frac{dv}{ds} = \sin\theta \end{cases} \Rightarrow d(\ln \sin\theta) = \frac{\cos\theta}{\sin\theta} d\theta = -\frac{f'}{f} du = -d(\ln f)$

$\Rightarrow f \sin\theta = \text{常数}$

15. $I = (1+f'^2) du^2 + u^2 dv^2$ 设 $C: r(s)$ 是曲线 (弧长参数)

则 $\Rightarrow$ Liouville 公式, $\begin{cases} \frac{du}{ds} = \frac{\cos\theta}{\sqrt{1+f'^2}} & (1) \\ \frac{dv}{ds} = \frac{\sin\theta}{u} & (2) \\ \frac{d\theta}{ds} = -\frac{\sin\theta}{u\sqrt{1+f'^2}} & (3) \end{cases}$

$\frac{(2)}{(1)} \Rightarrow \frac{dv}{du} = \frac{\sqrt{1+f'^2}}{u} \tan\theta$, $\frac{(3)}{(1)} \Rightarrow \frac{d\theta}{du} = -\frac{\tan\theta}{u}$

$u \sin\theta = c \text{ 常数} \Rightarrow \cos\theta = \sqrt{1 - \frac{c^2}{u^2}}$, $\tan\theta = \frac{c}{\sqrt{u^2 - c^2}}$

$$\frac{dv}{du} = \frac{C\sqrt{H}f^2}{u\sqrt{u^2-C^2}} \Rightarrow v = v_0 + \int_{v_0}^u \frac{C\sqrt{H}f^2}{u\sqrt{u^2-C^2}} du$$

$C=0$ 时, C^u 是线.

$$16. \quad K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}} \rightarrow (\sqrt{G})_{uu} = -K\sqrt{G}$$

$$G_u = 2\sqrt{G}(\sqrt{G})_u, \quad G_{uu} = 2((\sqrt{G})_u)^2 + 2\sqrt{G}(\sqrt{G})_{uu}$$

$$G_{uu}(0, v) = 2(\sqrt{G})_{uu}(0, v) = -2K(0, v).$$

将 $G(u, v)$ 在 $u=0$ 展开:

$$\begin{aligned} G(u, v) &= G(0, v) + u G_u(0, v) + \frac{u^2}{2} G_{uu}(0, v) + o(u^2) \\ &= 1 - u^2 K(0, v) + o(u^2) \end{aligned}$$

$$17. \quad I = du^2 + Gdv^2 \quad G(0, v) = 1 \quad u=0 \text{ 是垂直地线,}$$

$$0 = K_g(C_{u0}) = \frac{1}{2\sqrt{G(0, v)}} \frac{G_u(0, v)}{G(0, v)} \Leftrightarrow G_u(0, v) = 0.$$

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}} = \text{常数} \Leftrightarrow (\sqrt{G})_{uu} + K\sqrt{G} = 0$$

$$(1) \quad K=0. \quad (\sqrt{G})_{uu}=0 \Rightarrow (\sqrt{G})_u = \frac{G_u}{2\sqrt{G}} = f(v) \xrightarrow{G_u(0, v)=0} f(v)=0 \\ \Rightarrow \sqrt{G} = g(v) \xrightarrow{G(0, v)=1} g(v)=1 \Rightarrow G=1 \Rightarrow I = du^2 + dv^2$$

$$(2) \quad K = \frac{1}{a^2} > 0. \quad (\sqrt{G})_{uu} + \frac{1}{a^2}\sqrt{G} = 0 \Rightarrow \sqrt{G} = f(v) \cos \frac{u}{a} + g(v) \sin \frac{u}{a} \xrightarrow{G(0, v)=1} f(v)=1 \\ G = \left(\cos \frac{u}{a} + g(v) \sin \frac{u}{a} \right)^2 \Rightarrow G_u = 2 \left(\cos \frac{u}{a} + g(v) \sin \frac{u}{a} \right) \left(-\frac{1}{a} \sin \frac{u}{a} + \frac{1}{a} g(v) \cos \frac{u}{a} \right) \\ \xrightarrow{G(0, v)=1} g(v)=0 \Rightarrow G = \cos^2 \frac{u}{a} \\ \Rightarrow I = du^2 + \cos^2 \frac{u}{a} dv^2 = du^2 + \cos^2(\sqrt{K}u) dv^2$$

$$(3) \quad K = -\frac{1}{a^2} > 0. \quad (\sqrt{G})_{uu} - \frac{1}{a^2}\sqrt{G} = 0 \Rightarrow G = \cosh^2 \frac{u}{a} = \cosh^2(\sqrt{K}u)$$

$$\Rightarrow I = du^2 + \cosh^2(\sqrt{K}u) dv^2.$$

18. 取 P 为原点的局部大地坐标 (ρ, θ) . $I = d\rho^2 + G d\theta^2$, $\lim_{\rho \rightarrow 0} \sqrt{G} = 0$, $\lim_{\rho \rightarrow 0} (\sqrt{G})_{\rho} = 1$.

$$K = -\frac{\sqrt{G}_{\rho\rho}}{\sqrt{G}} \Leftrightarrow \sqrt{G}_{\rho\rho} = -K\sqrt{G} \Rightarrow \lim_{\rho \rightarrow 0} (\sqrt{G})_{\rho\rho} = 0$$

$$\sqrt{G}_{\rho\rho\rho} = -K_{\rho}\sqrt{G} - K(\sqrt{G})_{\rho\rho} \Rightarrow \lim_{\rho \rightarrow 0} (\sqrt{G})_{\rho\rho\rho} = -K(\Phi)$$

将 $\sqrt{G}$ 在 $\rho \rightarrow 0^+$ 展开:

$$\begin{aligned} \sqrt{G} &= \sqrt{G(0^+, \theta)} + \rho (\sqrt{G(0^+, \theta)})_{\rho} + \frac{\rho^2}{2} (\sqrt{G(0^+, \theta)})_{\rho\rho} + \frac{\rho^3}{6} (\sqrt{G(0^+, \theta)})_{\rho\rho\rho} + o(\rho^3) \\ &= \rho - \frac{\rho^3}{6} K(\Phi) + o(\rho^3) \end{aligned}$$

$$\hookrightarrow L_r = \int_0^{2\pi} \sqrt{G(r, 0)} d\theta = \int_0^{2\pi} \left(r - \frac{r^3}{6} K(\Phi) + o(r^3) \right) d\theta = 2\pi \left(r - \frac{r^3}{6} K(\Phi) + o(r^3) \right)$$

$$K(\Phi) = \lim_{r \rightarrow 0^+} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}$$

$$\begin{aligned} \hookrightarrow A_r &= \int_0^r \int_0^{2\pi} \sqrt{G(\rho, 0)} d\theta d\rho = \int_0^r \int_0^{2\pi} \left(\rho - \frac{\rho^3}{6} K(\Phi) + o(\rho^3) \right) d\theta d\rho \\ &= \int_0^r (2\pi(\rho - \frac{\rho^3}{6} K(\Phi)) + o(\rho^3)) d\rho = \pi r^2 - \frac{\pi}{12} r^4 K(\Phi) + o(r^4) \end{aligned}$$

$$K(\Phi) = \lim_{r \rightarrow 0^+} \frac{12}{\pi} \frac{\pi r^2 - A_r}{r^4}$$

19. 测地圆 $C_{\rho_0}: r(\rho, \theta)$.

$$\text{由 P.127, } I = \begin{cases} d\rho^2 + \rho^2 d\theta^2, & K=0 \\ d\rho^2 + \frac{1}{K} \sin^2(K\rho) d\theta^2, & K>0 \\ d\rho^2 - \frac{1}{K} \sinh^2(K\rho) d\theta^2, & K<0 \end{cases}$$

$$\hookrightarrow \text{Liouville 公式, } k_g(C_{\rho_0}) = \begin{cases} \frac{1}{\rho_0}, & K=0 \\ \frac{\sqrt{K} \cosh(K\rho_0)}{\sinh(K\rho_0)}, & K>0 \\ \frac{\sqrt{K} \cosh(K\rho_0)}{\sinh(K\rho_0)}, & K<0 \end{cases}$$

20. $\forall p \in S$, 取 p 附近的取参数 (u, v) 使得 u -线是一族测地线.

$$u \mid I = E du^2 + G dv^2. \Rightarrow k_g(C_v) = -\frac{1}{\sqrt{G}} \cdot \frac{E_v}{E} = 0, E_v = 0.$$

故 $E(u, v) = E(u)$. 设另一族测地线 v -线夹角为 $\theta_0 \in (0, \pi)$.

对过 p 的测地线 C_p 应用 Liouville 公式,

$$\begin{aligned} 0 &= k_g(C_p) = -\frac{1}{2\sqrt{G}} \frac{E_v}{E} \cos \theta_0 + \frac{1}{2\sqrt{E}} \frac{G_u}{G} \sin \theta_0 \\ &= \frac{1}{2\sqrt{E}} \frac{G_u}{G} \sin \theta_0. \end{aligned}$$

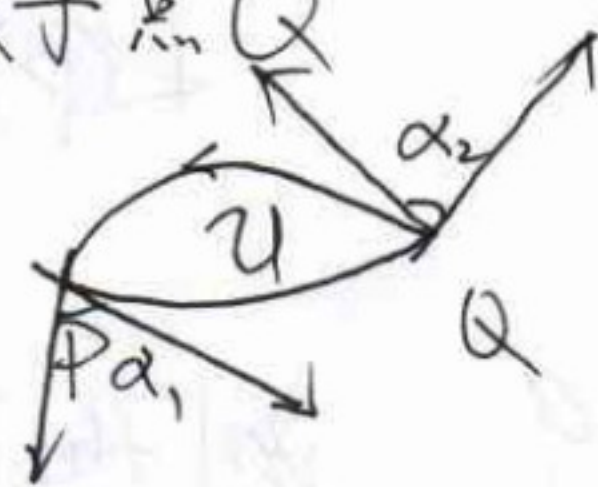
故 $G_u = 0$. $G(u, v) = G(v)$. 作变换 $\tilde{u} = \int \sqrt{E} du, \tilde{v} = \int \sqrt{G} dv$.

$$u \mid I = E du^2 + G dv^2 = d\tilde{u}^2 + d\tilde{v}^2.$$

故 $S \sim$ 等距平面.

21. 假设由 p 出发的两条测地线相交于点 Q

考虑如图: 二边形. $\Rightarrow$ Gauss-Bonnet 公式.



$$\iint_U K dA + \alpha_1 + \alpha_2 = 2\pi, \quad \alpha_i \in (0, \pi)$$

故 $K < 0$
 $2\pi > \alpha_1 + \alpha_2 = 2\pi - \iint_U K dA > 2\pi$, 矛盾.

22. $\Rightarrow$ Gauss-Bonnet 公式, (β_i 是外角)

$$\iint_A K d\sigma + \int_{\partial A} k_g ds + \sum_{i=1}^4 \beta_i = 2\pi$$

$$\nabla \alpha_i + \beta_i = \pi, \quad \iint_A K d\sigma + \int_{\partial A} k_g ds = \sum_{i=1}^4 \alpha_i - 2\pi.$$

23. $\nabla d\tilde{s}^2 = d\tilde{u}^2 + e^{\frac{2\tilde{u}}{a}} d\tilde{v}^2 = e^{\frac{2\tilde{u}}{a}} (e^{-\frac{2\tilde{u}}{a}} d\tilde{u}^2 + d\tilde{v}^2)$

(作变换 $\begin{cases} u = \frac{1}{a}\tilde{u} \\ v = e^{-\frac{\tilde{u}}{a}} \end{cases}, u, v) = \frac{a^2}{v^2} (du^2 + dv^2).$

24. " $\Rightarrow$ ". C : 单位切向量场 $\vec{T}(s) = \vec{v}(t) \cdot \frac{dt}{ds}$.

$$\begin{aligned} C \text{ 是测地线} &\Leftrightarrow \vec{0} = \frac{D\vec{T}(s)}{ds} = \frac{dt}{ds^2} \vec{T}(s) + \frac{dt}{ds} \frac{D\vec{v}(t)}{dt} \\ &= \frac{dt}{ds} \left(\frac{d^2t}{ds^2} \vec{T}(t) + \frac{dt}{ds} \frac{D\vec{v}(t)}{dt} \right) \end{aligned}$$

取 $\lambda = -\frac{d^2t}{ds^2} \cdot \frac{ds}{dt}$, 则 $\frac{D\vec{v}(t)}{dt} + \lambda \vec{v}(t) = 0$.

" $\Leftarrow$ ". $\vec{v}(t) = \vec{T}(s) \frac{ds}{dt}$.

$$\begin{aligned} \vec{0} = \frac{D\vec{v}(t)}{dt} + \lambda \vec{v}(t) &= \frac{ds}{dt^2} \vec{T}(s) + \frac{ds}{dt} \frac{ds}{dt} \frac{D\vec{T}(s)}{ds} + \lambda \frac{ds}{dt} \vec{T}(s) \\ &= \left(\frac{ds}{dt} \right)^2 \frac{D\vec{T}(s)}{ds} + \left(\lambda \frac{ds}{dt} + \frac{d^2s}{dt^2} \right) \vec{T}(s) \end{aligned}$$

取 $\vec{n}(s)$ 是沿 C 的单位切向量场, 使得 $\vec{n} \perp \vec{T}$ 且 $\vec{T}, \vec{n}$ 与 $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ 同向.

则 $\langle \vec{n}(s), \cdot \rangle \Rightarrow 0 = \left(\frac{ds}{dt} \right)^2 k_g \Rightarrow k_g = 0$.